Todorova, Tamara

Book Part
Advanced Differential and Difference Equations


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Chapter 11. Advanced Differential and Difference Equations

In this chapter we deal with harder differential and difference equations. We already discussed first-order equations in which a first-order derivative or difference is involved. Some more sophisticated cases are second-, third-, or higher-order differential or difference equations. The chapter is split in two: first we cover more advanced differential equations, and then we turn onto their discrete-time counterpart, higher-order difference equations.

Second-order Differential Equations

Consider the linear differential equation
\[ y^{(n)}(t) + u_1(t)y^{(n-1)}(t) + \ldots + u_{n-1}(t)y'(t) + u_n(t)y(t) = v(t) \]

Since it contains the \( n^{th} \) derivative \( y^n(t) \) of the function \( y(t) \), it is an \( n \)-th order differential equation with variable coefficients. It is easy to notice that when only a first-derivative \( y'(t) \) is involved, the equation becomes the special case of a first-order differential equation
\[ \frac{dy}{dt} + u(t)y = v(t) \]

which we are already familiar with. By analogy with the constant coefficient case, we have the general linear \( n^{th} \) order equation
\[ y^{(n)}(t) + a_1y^{(n-1)}(t) + \ldots + a_{n-1}y'(t) + a_ny = b \]

where again the \( n^{th} \) derivative \( y^{(n)}(t) \) is involved; but this time, the functions \( u_i(t) \) and \( v(t) \) \((i = 1, 2, \ldots, n)\) correspond to the constants \( a_i \) and \( b \), respectively. Similar to the first-order equation case when only the first derivative \( y'(t) \) is involved, we have the familiar equation
\[ \frac{dy}{dt} + ay = b \]

We found the general solution to this simple first-order differential equation to be
\[ y(t) = y_c + y_p = A e^{-at} + \frac{b}{a} \]

where \( y_p = \frac{b}{a} \) is the particular integral giving the intertemporal equilibrium. This implies that we have the simplest possible type of solution for \( y(t) \); that is, \( y(t) = c \) where the function \( y \) is constant in time and the derivative \( \frac{dy}{dt} \) is zero. Consider now the case
\[ y''(t) + a_1y'(t) + a_2y = b \]

where the highest derivative is the second-order derivative \( y''(t) \). If we again assume the simplest possible type, that is, \( y' \) being a constant, we should have
\[ y''(t) = y'(t) = 0 \]
and the particular integral is
\[ y_p = \frac{b}{a_2} \quad a_2 \neq 0 \]

**Example:** Find the particular integral of the equation \( y''(t) + y'(t) - 2y = -6 \). Since \( a_2 = -2 \) and \( b = -6 \), substituting in the expression for the particular integral yields \( y_p = \frac{-6}{-2} = 3 \).

What if \( a_2 = 0 \) so the expression for the particular integral is undefined? Then it must be that \( y \) is no longer constant. A simple case to consider is \( y = ct \) where again \( c = \text{const} \). Then the differential equation becomes

\[ y''(t) + a_1 y'(t) = b \quad a_2 = 0 \]

Since \( y = ct \), it follows that \( y'(t) = c \) and \( y''(t) = 0 \), which reduces the equation to

\[ y'(t) = \frac{b}{a_1} \]

We find the particular integral by integrating \( y'(t) \) with respect to \( t \), which gives

\[ y_p = \frac{b}{a_1}t \quad a_2 = 0 \quad a_1 \neq 0 \]

Given that this time \( y_p \) is a nonconstant function of time, it constitutes a moving equilibrium.

In the case when \( a_1 = a_2 = 0 \), the second-order differential equation becomes

\[ y''(t) = b \]

Integrating \( y''(t) \) twice with respect to \( t \) gives

\[ y_p = \frac{bt^2}{2} \quad a_1 = a_2 = 0 \]

**The Complementary Function**

In the case of the first-order linear differential equation, its complementary function was the general solution of the homogeneous (reduced) equation \( y'(t) + ay(t) = 0 \), i.e., \( y(t) = Ae^{-at} \). Generally, an expression of the form \( Ae^{rt} \) fits well into complementary functions. One reason why we can apply this exponential term to a second-order differential equation is that the latter is a second-order generalization of the first-order homogeneous equation. If we assume the solution for the function \( y(t) \) to be of an exponential type \( y(t) = Ae^{rt} \), then we have

\[ y'(t) = rAe^{rt} \quad \text{and} \quad y''(t) = r^2Ae^{rt} \]

Substituting these values of the derivatives and the parental function in the homogeneous second-order differential equation yields

\[ r^2Ae^{rt} + arAe^{rt} + a_2Ae^{rt} = 0 \]

which gives rise to the characteristic equation \( r^2 + ar + a_2 = 0 \) and the two characteristic roots

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \]
where by Viete’s formula\(^1\) \( r_1 + r_2 = -a_1 \) and \( r_1 r_2 = a_2 \). These two roots result in two solutions for 
\[ y(t) = A e^{r_1 t} \quad \text{and} \quad y_2 = A e^{r_2 t} \]
where \( A_1 \) and \( A_2 \) are two arbitrary constants and the complementary function of the nonhomogeneous (complete) equation is \( y_c = y_1 + y_2 \). Three possible situations exist in relation to the characteristic roots \( r_1 \) and \( r_2 \).

**Case 1. Distinct real roots**

If \( a_1^2 > 4a_2 \), then both roots \( r_1 \) and \( r_2 \) are distinct real numbers and we can write
\[ y_c = y_1 + y_2 = A_1 e^{r_1 t} + A_2 e^{r_2 t} \]
For particular values of the two constants \( A_1 \) and \( A_2 \) implied by some initial conditions of \( y(t) \) and its derivatives, we can find the general solution to the complete equation as the sum of the complementary function and the particular integral
\[ y(t) = y_c + y_p = y_1 + y_2 + y_p = A_1 e^{r_1 t} + A_2 e^{r_2 t} + y_p \]

**Example:** Solve the differential equation \( y''(t) + y'(t) - 2y = -6 \). We already found the particular integral of this nonhomogeneous equation to be \( y_p = 3 \). How to find the complementary function?

We see that the equation fits this first case since \( a_1 = 1 \), \( a_2 = -2 \) and \( a_1^2 > 4a_2 \) because \( 1 > -8 \). Furthermore, the characteristic roots are
\[ r_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \]
\[ r_1 = 1 \quad \text{and} \quad r_2 = -2 \]
\[ y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + y_p = A_1 e^t + A_2 e^{-2t} + 3 \]
To find the particular values of the constants \( A_1 \) and \( A_2 \), we need two initial conditions. Suppose \( y(0) = 10 \) and \( y'(0) = -2 \) where the initial moment is \( t = 0 \). Substituting for \( t = 0 \) we obtain
\[ y(0) = A_1 e^0 + A_2 e^{-2(0)} + 3 = A_1 + A_2 + 3 = 10 \]
Differentiating \( y(t) \) with respect to \( t \), we get
\[ y'(t) = A_1 e^t - 2A_2 e^{-2t} \]
Then at \( t = 0 \),
\[ y'(0) = A_1 e^0 - 2A_2 e^{-2(0)} = A_1 - 2A_2 = -2 \]
which leads to the system of equations
\[ A_1 + A_2 = 7 \]
\[ A_1 - 2A_2 = -2 \]
with solutions \( A_1 = 4 \) and \( A_2 = 3 \). Substituting to obtain the definite solution of the second-order differential equation,

\(^1\) Named after the French mathematician François Viete (1540-1603).
Case 2. Single real root

If \( a_1^2 = 4a_2 \), there is only one root (also called a coincident or repeated root) \( r = -\frac{a_1}{2} \). Then the complementary function is

\[ y_c = A_1e^{rt} + A_2h(t) \]

where \( h(t) \) is a function that cannot be a constant multiple of \( e^{rt} \). Therefore, we set \( h(t) = te^{rt} \), and the general solution to the second-order differential equation is

\[ y(t) = y_c + y_p = A_1e^{rt} + A_2te^{rt} + y_p \]

Example: Solve the differential equation \( y''(t) + 2y'(t) + y = 0 \). We can easily notice that \( a_1^2 = 4a_2 \) since \( a_1 = 2, a_2 = 1 \) and \( 2^2 = 4(1) \). Thus, the example is one of a single real root \( r = -\frac{a_1}{2} = -\frac{2}{2} = -1 \) and \( y_p = 5 \). Therefore, \( y(t) = y_c + y_p = A_1e^{-t} + A_2te^{-t} + 5 \).

Case 3. Complex roots

What if \( a_1^2 < 4a_2 \)? Then the roots \( r_1 \) and \( r_2 \) contain the square root of a negative number \( i = \sqrt{-1} \) called an imaginary number. The very roots are called complex numbers as they contain a real part and an imaginary part, for instance, \( (-5 + i) \), where we already defined \( i \). Complex numbers cannot be ordered along the real line and, therefore, do not belong to the real-number system. They can generally be represented in the form \((m + ni)\) where \( m \) and \( n \) are two real numbers. A complex number can be represented graphically in the \( xy \)-plane where \( x \) is the real-number axis and \( y \) is the imaginary-number axis. In this two-dimensional diagram known as the Argand diagram (shown by Figure 1) \( m \) is plotted on the horizontal axis and \( n \) on the vertical. Thus when \( n = 0 \), the complex number does not have an imaginary part and reduces to a real one. When \( m = 0 \), it is solely an imaginary number. By Pythagoras theorem the length of the \( ON \) line is found as the radius vector

\[ R = \sqrt{m^2 + n^2} \]

![Argand diagram](image)

When \( a_1^2 < 4a_2 \), the two roots of the characteristic equation are a pair of conjugate complex numbers:

\[ r_{1,2} = m \pm ni \quad \text{where} \quad m = -\frac{a_1}{2} \quad n = \frac{\sqrt{4a_2 - a_1^2}}{2} \quad \text{and} \quad i = \sqrt{-1} \]

In the complex-root case the complementary function of the differential equation becomes
Example: Find the roots of the characteristic equation \( r^2 + r + 2 = 0 \). Express the complementary function for this equation. We obtain a pair of conjugate complex numbers for the two characteristic roots.

\[
r_{1,2} = -\frac{1 \pm \sqrt{7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i
\]

It can easily be checked that, in accordance with Viete’s formula, \( r_1 + r_2 = -a_1 = -1 \) and \( r_1r_2 = a_2 = 2 \).

Since \( m = -\frac{1}{2} \) and \( n = \frac{\sqrt{7}}{2} \), the complementary function is

\[
y_c = e^{-\frac{t}{2}} \left( A_1 e^{\frac{\sqrt{7}it}{2}} + A_2 e^{-\frac{\sqrt{7}it}{2}} \right)
\]

where the imaginary number \( i \) appears in the exponents of the two expressions of the complementary function. To understand such imaginary exponential functions better, we should transform them into circular functions, which requires some discussion of trigonometry.

**An Excursion into Trigonometric Functions**

In the next part we will briefly revise some basics of trigonometric functions that may be familiar to you from high school. Trigonometric functions are often connected with complex numbers. Given an angle \( \theta \), as shown in Figure 2 depicting a circle with a radius \( R \), the trigonometric functions are

\[
\sin \theta = \frac{n}{R} \quad \cos \theta = \frac{m}{R}
\]

where \( m, n \) and \( R \) happen to be sides of the right-angle triangle \( OMN \). Two more trigonometric functions can be defined on the basis of these two original functions:

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{n}{m} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{m}{n}
\]

![Figure 2](image_url)

The angle \( \theta \) is measured in degrees (say, 90°) or in radians which allow expressing the derivatives of trigonometric functions more easily. The size of the angle \( \theta \) is defined by the \( PN \) arc. A complete
circle like $PQST$ involves an angle of $2\pi$ radians which is exactly $360^\circ$ or $\pi = 180^\circ$. Thus, radians transform into degrees according to the following conversion table (see Table 1).

<table>
<thead>
<tr>
<th>Degrees</th>
<th>0</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>135°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
</tr>
</thead>
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<tr>
<td>Radians</td>
<td>$0$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{3\pi}{4}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>$\sin \theta$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1

The sine and the cosine functions are periodic and repeat every $360^\circ$. They both fluctuate between 0 and 1 but differ in their peaks as shown on the following two diagrams in Figures 3a and 3b.

![Figure 3a](image1)

![Figure 3b](image2)

The sine and cosine functions have the following properties:

$\sin(-\theta) = -\sin \theta$
$\cos(-\theta) = \cos \theta$
$\sin^2 \theta + \cos^2 \theta = 1$
$\sin(\theta_1 \pm \theta_2) = \sin \theta_1 \cos \theta_2 \pm \cos \theta_1 \sin \theta_2$
$\cos(\theta_1 \pm \theta_2) = \cos \theta_1 \cos \theta_2 \pm \sin \theta_1 \sin \theta_2$

Given that both $\sin \theta$ and $\cos \theta$ are continuous and smooth, they are differentiable. The derivatives of the functions, applicable to radians only, turn out to be
\[
\frac{d\sin \theta}{d\theta} = \cos \theta \quad \frac{d\cos \theta}{d\theta} = -\sin \theta \quad \frac{d\tan \theta}{d\theta} = \frac{1}{\cos^2 \theta}
\]

In the general case, given that \( u \) is a differentiable function of \( x \), the derivatives for sine and cosine can be written as

\[
\frac{d\sin u}{dx} = \cos u \frac{du}{dx} \quad \frac{d\cos u}{dx} = -\sin u \frac{du}{dx}
\]

**Example:** Find the second derivative of \( \cos \theta \). Applying the formula twice, we get

\[
\frac{d^2 \cos \theta}{d\theta^2} = \frac{d(\sin \theta)}{d\theta} = -\cos \theta
\]

**Example:** Find the derivative of the trigonometric function \( \sin(5x^2 + 2) \). Using the general formula,

\[
\frac{d\sin(5x^2 + 2)}{dx} = 10x \cos(5x^2 + 2)
\]

**Transforming Complex Numbers into Trigonometric Functions**

As long as the two Cartesian coordinates \( m \) and \( n \) are defined, we can find the angle \( \theta \) and the radius \( R \), also known as polar coordinates. A basic relationship between Cartesian and polar coordinates we obtained previously is \( R = \sqrt{m^2 + n^2} \). In the opposite case knowing the values of \( R \) and \( \theta \), we can write \( m = R \cos \theta \) and \( n = R \sin \theta \). Thus, the pair of conjugate complex numbers \( m \pm ni \) becomes

\[
m \pm ni = R \cos \theta \pm Ri \sin \theta = R(\cos \theta \pm i \sin \theta)
\]

By what is known as Euler’s formula for complex numbers, which we will not prove here,

\[
e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta \quad \text{so}
\]

\[
m \pm ni = R(\cos \theta \pm i \sin \theta) = Re^{i\theta}
\]

Abraham De Moivre\(^3\) discovered further that

\[
[R(\cos \theta \pm i \sin \theta)]^k = R^k (\cos k\theta \pm i \sin k\theta)
\]

This result allows to find a pair of conjugate complex numbers raised to the power \( k \) such as \((m \pm ni)^k\) where by De Moivre’s theorem

\[
(m \pm ni)^k = R^k e^{\pm ik\theta} = R^k (\cos k\theta \pm i \sin k\theta)
\]

Going back to the complementary function \( y_c = e^{nit} (A_1 e^{nint} + A_2 e^{-nint}) \), we let \( \theta = nt \) and by Euler’s formula

\[
e^{nit} = \cos nt + i \sin nt \quad \text{and} \quad e^{-nint} = \cos nt - i \sin nt
\]

and substituting these consequently into the complementary function,

\[
y_c = e^{nit} \left[ A_1 (\cos nt + i \sin nt) + A_2 (\cos nt - i \sin nt) \right] = e^{nit} (B_1 \cos nt + B_2 \sin nt)
\]

where \( B_1 = A_1 + A_2 \) and \( B_2 = (A_1 - A_2) i \)

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\(^2\) Named after the talented French mathematician Rene Descartes (1596-1650).

\(^3\) Except relating complex numbers to trigonometry, the French mathematician Abraham De Moivre (1667-1754) is credited for the study of normal distribution and probability theory.
Example: Find the polar and exponential forms of \((\sqrt{2} + \sqrt{2}i)\). The Cartesian coordinates are 

\[ m = \sqrt{2} \quad \text{and} \quad n = \sqrt{2}. \]

We can find \( R \) by the formula \( R = \sqrt{m^2 + n^2} = \sqrt{2 + 2} = 2 \). Thus, we can find the sine and cosine functions as

\[
\sin \theta = \frac{n}{R} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos \theta = \frac{m}{R} = \frac{\sqrt{2}}{2}
\]

But we know that these are the values for \( \theta = \frac{\pi}{4} = 45^\circ \). Hence, from the formula

\[ m \pm ni = R(\cos \theta \pm i \sin \theta) = R e^{\pm i \theta} \]

\[ \sqrt{2} + \sqrt{2}i = 2 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 2e^{i\pi/4} \]

Example: Find the complementary function and the particular integral of the differential equation

\[ y''(t) + 2y'(t) + 10y = 20 \]

for which the initial conditions are \( y(0) = 3 \) and \( y'(0) = 11 \). Here we have \( a_1 = 2, a_2 = 10 \) and \( b = 20 \), so for the particular integral we get \( y_p = \frac{b}{a_2} = \frac{20}{10} = 2 \). Furthermore, since \( a_1^2 < 4a_2 \) or \( 4 < 40 \), the characteristic roots are \( r_{1,2} = m \pm ni \) and

\[ m = -\frac{a_1}{2} = -1 \quad \text{and} \quad n = \sqrt{\frac{4a_2 - a_1^2}{2}} = \sqrt{\frac{4(10) - 2^2}{2}} = 3 \]

\[ y_c = e^{mt} (B_1 \cos nt + B_2 \sin nt) = e^{-t} (B_1 \cos 3t + B_2 \sin 3t) \]

Thus, the general solution of the differential equation is

\[ y(t) = y_c + y_p = e^{-t} (B_1 \cos 3t + B_2 \sin 3t) + 2 \]

To definitize the constants \( B_1 \) and \( B_2 \) we must use the initial conditions. Substituting for \( t = 0 \) in \( y(t) \), we obtain \( y(0) = e^0 (B_1 \cos 0 + B_2 \sin 0) + 2 = 3 \) where we know that \( \cos 0 = 1 \) and \( \sin 0 = 0 \), so we have \( B_1 + 2 = 3 \) or \( B_1 = 1 \). Expressing \( y'(t) \) from \( y(t) \),

\[ y'(t) = -e^{-t} (B_1 \cos 3t + B_2 \sin 3t) + e^{-t} (-3B_1 \sin 3t + 3B_2 \cos 3t) \]

and setting \( t = 0 \) again

\[ y'(0) = -e^0 (B_1 \cos 0 + B_2 \sin 0) + e^0 (-3B_1 \sin 0 + 3B_2 \cos 0) = -B_1 + 3B_2 = 11 \quad \text{or} \quad -1 + 3B_2 = 11 \]

\[ 3B_2 = 12 \]

\[ B_2 = 4 \]

So, the differential equation is

\[ y(t) = e^{-t} (\cos 3t + 4 \sin 3t) + 2 \]

**Dynamic Stability**

The time path of the complementary function \( y_c = e^{mt} (B_1 \cos nt + B_2 \sin nt) \) depends on the sine and cosine functions as well as on the term \( e^{mt} \). Since the period of the trigonometric functions is \( 2\pi \) and
their amplitude is 1, their graphs will repeat their shape every time the expression $nt$ increases by $2\pi$. Alternatively,

$$\theta = nt = nt + 2\pi = n\left(t + \frac{2\pi}{n}\right)$$

The first term in the parentheses $B_1 \cos nt$ is a cosine function of $t$ with a period $\frac{2\pi}{n}$. Similarly, the second term $B_2 \sin nt$ has the same period and fluctuates between $-B_2$ and $B_2$. The dynamic stability of the function $y(t)$ depends solely on the third term, $e^{mt}$ such that for a positive $m$, as $t \to \infty$, the amplitude of $(B_1 \cos nt + B_2 \sin nt)$ magnifies and causes an explosive fluctuation for $y(t)$.

If $m = 0$, the complementary function has a uniform fluctuation. When $m$ is negative, the time path of the function $y(t)$ is dynamically stable (See Figures 4a, 4b, and 4c).

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**Figure 4**

(a)Explosive fluctuation

(b)Uniform fluctuation

(c)Damped fluctuation
For the second-order differential equation
\[ y''(t) + a_1 y'(t) + a_2 y = b \]
we found that the time path of the parental function \( y(t) \) in the case of distinct real roots is
\[ y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + y_p \]
where \( A_1 \) and \( A_2 \) are arbitrary constants. The dynamic stability of the function is insured only if both roots \( r_1 \) and \( r_2 \) are negative; so, as \( t \to \infty \), \( y(t) \) converges to its equilibrium level \( y_p \). If even one of the roots is positive, its exponential term becomes infinitely large, thereby precluding convergence.

With a single root, the solution is
\[ y(t) = A_1 e^{rt} + A_2 te^{rt} + y_p \]
Here the necessary and sufficient condition for dynamic stability is that the single root \( r \) be negative as \( t \to \infty \). The second multiplicative term, \( A_2 te^{rt} \), also approaches zero because with a negative exponent the exponential term reaches zero faster than \( t \) grows.

Finally, in the case of complex roots, the solution is
\[ y(t) = e^{mt} (A_1 e^{nit} + A_2 e^{-nit}) + y_p \]
where
\[ r_{1,2} = m \pm ni \quad m = -\frac{a_1}{2} \quad \text{and} \quad n = \frac{\sqrt{4a_2 - a_1^2}}{2} \]
The condition for convergence of the \( y(t) \) path is \( m < 0 \), that is, the real part \( m \) of the complex roots to be negative. Then, for the three cases, it is enough to demand that the real part of every characteristic root be negative to ensure dynamic stability of equilibrium.

**The Arrow-Pratt Measure of Risk Aversion**

Let a person have wealth in the amount \( w \) and \( u(w) \) be the utility function over this wealth. To measure the concavity of the utility function \( u(w) \) in portfolio choice theory, Kenneth Arrow and John W. Pratt use the so-called Arrow-Pratt measure of relative (or absolute) risk aversion at wealth level \( w \) generally given by the expression
\[ E_{uu} = -\frac{u''(w)w}{u'(w)} \]
which is nothing but the elasticity of the marginal utility function \( u'(w) \) with respect to the wealth level \( w \). Since we do not want total utility of wealth to be declining, we require \( u'(w) > 0 \). If we assume the individual to have a constant relative risk aversion (say, be either risk averse or risk loving or risk neutral), we can adopt a constant elasticity of \( k \). Thus, the expression becomes
\[ -\frac{u''(w)w}{u'(w)} = k \]
which gives the second-order differential equation
\[ u''(w) + k \frac{w}{u'(w)} = 0 \]
Furthermore, if we substitute \( u_w = u'(w) \) for the marginal utility of wealth,
\[ u_w' + k \frac{w}{u_w} = 0 \]
we obtain a first-order differential equation in marginal utility \( u_w \). Rearranging and solving by the separation of variables method,
\[ \frac{u'_w}{u_w} = -\frac{k}{w} \]

\[ \int \frac{u'_w}{u_w} \, dw = -\int \frac{k}{w} \, dw \]

\[ \ln u_w = -k \ln w + c \quad \text{and taking the antilog of both sides,} \]

\[ u_w = e^{-k \ln w} \cdot e^c = C w^{-k} \]

Integrating marginal utility \( u_w = u'(w) \) further to obtain the total utility function, we have

\[ u(w) = \int u'(w) \, dw = \int C w^{-k} \, dw = \begin{cases} C \frac{w^{1-k} + c_1}{1-k} & \text{for } k \neq 1 \\ C \ln w + c_2 & \text{for } k = 1 \end{cases} \]

A special case of the measure of risk aversion is the Arrow-Pratt measure of absolute risk aversion representing the percent rate of change of marginal utility of wealth \( u'(w) \) at wealth level \( w \). It is again a measure of the concavity of the total utility function \( u(w) \) and is given by the expression

\[ -\frac{u''(w)}{u'(w)} = a(w) \]

thus giving rise to the second-order differential equation

\[ u''(w) + a(w)u'(w) = 0 \]

which can be solved for a specific function \( a(w) \).

**Market Equilibrium with Price Expectations**

Very often, market participants base their demand and supply decisions on their expectations about the price and its behavior in the future. Those expectations are often influenced not only by the price prevailing at the moment, but also by the trends in the price movements. We can apply second-order differential equations to establish the time path of market price assuming equilibrium in each moment in time. Let us take, for example, that

\[ q_d = \alpha - \beta p + up' + \alpha p^* \quad \alpha, \beta > 0 \]

\[ q_s = -\gamma + \delta p \quad \gamma, \delta > 0 \]

where \( p' = \frac{dp}{dt} \) and \( p^* = \frac{d^2 p}{dt^2} \). In the context of price trends, a positive \( p' \) implies that market price is rising and a positive \( p^* \) shows that it is rising at an increasing rate.

Then, if \( u > 0 \), a rising price increases market demand. Buyers, expecting price to rise, would prefer to increase current consumption. An example of such move is the real estate market in Bulgaria. Prior to Bulgaria’s joining the European Union, people expected the prices of houses to continue to rise, so they increased their purchases, thus pushing the prices further up. Conversely, when \( u < 0 \), people expect the price trend to reverse and, therefore, they cut back on their purchases in expectation of a lower price in the future. Similarly, the Bulgarian real estate market experienced a slowdown in housing prices after the country was accepted into the EU. Furthermore, the global financial crisis influenced the decisions of house buyers negatively. Expecting real estate prices to fall, they stopped buying, which contributed further to the decline of prices. Thus, this continuous-time model illustrates how people’s expectations of the future shape current prices. In their buying decisions, consumers
may be driven not only by the direction of change in market price, but also by the rate at which this change occurs given here by the parameter $v$.

Equating market demand with market supply, we get

$$q_d = q_s \text{ or } \alpha - \beta p + up' + vp^* = -\gamma + \delta p$$

which transforms into

$$vp^* + up' - (\beta + \delta)p = -\alpha - \gamma$$

Normalizing this second-order differential equation,

$$p^* + \frac{u}{v}p' - \frac{(\beta + \delta)}{v}p = -\frac{\alpha + \gamma}{v}$$

Note that the present model assumes market clearance at every moment in time. Thus, every price established in the market at any given moment is an equilibrium price, although this market equilibrium may not be the intertemporal equilibrium. You can think of the intertemporal equilibrium price as the normal price that should prevail in the market at any moment. Yet, at different moments the market might clear at too high or too low levels of price. When Bulgaria experiences a real estate bubble, prices of apartments are abnormally high – but demand and supply meet, so the market clears and the price is the market equilibrium price, though not the normal price level or the intertemporal equilibrium. In times of a deep recession, the market still clears – but at very low price levels, much below the intertemporal equilibrium.

For the second-order differential equation that obtains, we have

$$a_1 = \frac{u}{v} \quad a_2 = -\frac{\beta + \delta}{v} \quad \text{and} \quad b = -\frac{\alpha + \gamma}{v}$$

In order to solve the differential equation, we need to find the time path of the price function $p(t)$.

The particular integral will give us the intertemporal equilibrium price:

$$p_p = \frac{b}{a_2} = \frac{\alpha + \gamma}{\beta + \delta}$$

Note that the intertemporal equilibrium price is positive, as it should be, since all the constants in it are positive. Also, because it is constant, it represents a stationary, not a moving equilibrium. To find the complementary function, we discuss three cases.

**Case 1. Distinct real roots**

$$\left(\frac{u}{v}\right)^2 > 4\left(\frac{\beta + \delta}{v}\right)$$

The complementary function for this case is

$$p_c = A_1 e^{\theta t} + A_2 e^{\eta t}$$

where to definitize the constants $A_1$ and $A_2$, we need some initial conditions for price.

$$r_{1,2} = \frac{1}{2} \left[ -\frac{u}{v} \pm \sqrt{\left(\frac{u}{v}\right)^2 + 4 \left(\frac{\beta + \delta}{v}\right)} \right]$$

Thus, the general solution is
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\[ p(t) = p_c + p_p = A_1e^{rt} + A_2e^{st} + \frac{\alpha + \gamma}{\beta + \delta} \]

Case 2. Single real root

\[ \left( \frac{u}{v} \right)^2 = -4 \left( \frac{\beta + \delta}{v} \right) \]

The single root is \( r = -\frac{a_1}{2} = -\frac{u}{2v} \). Thus, the general solution becomes

\[ p(t) = A_1e^{\frac{ut}{2v}} + A_2e^{\frac{-ut}{2v}} + \frac{\alpha + \gamma}{\beta + \delta} \]

Case 3. Complex roots

\[ \left( \frac{u}{v} \right)^2 < -4 \left( \frac{\beta + \delta}{v} \right) \]

The characteristic roots are a pair of conjugate complex numbers \( r_{1,2} = m \pm ni \) where

\[ m = -\frac{a_1}{2} = -\frac{u}{2v} \quad \text{and} \quad n = \frac{\sqrt{4a_2 - a_1^2}}{2} = \frac{1}{2} \sqrt{-4 \left( \frac{\beta + \delta}{v} \right) - \left( \frac{u}{v} \right)^2} \]

For the general solution, we have

\[ p(t) = e^{mt} (A_1e^{nit} + A_2e^{-nit}) + \frac{\alpha + \gamma}{\beta + \delta} = e^{\frac{-ut}{2v}} (B_1 \cos nt + B_2 \sin nt) + \frac{\alpha + \gamma}{\beta + \delta} \]

If \( v > 0 \), then \(-4 \left( \frac{\beta + \delta}{v} \right)\) is always negative, so only the first case of distinct real roots is possible.

Under the square root we get a number bigger than \( \left( \frac{u}{v} \right)^2 \), which means that at least one characteristic root is positive. Therefore, the intertemporal equilibrium must be dynamically unstable. If \( v < 0 \), all three cases are possible. In the case of distinct real roots both roots will be negative, given \( u < 0 \). This is because the expression under the square root is definitely smaller than \( \left( \frac{u}{v} \right)^2 \) and the free term \( \frac{u}{v} \) is positive. Hence, both characteristic roots turn out to be negative. The condition \( u, v < 0 \) also ensures that the single root is negative. In the third case of complex numbers when \( u, v < 0 \), \( m \) turns out to be negative too. Therefore, the dynamic stability of the price function in each case is ensured when both parameters \( u \) and \( v \) are negative.

The Relationship between Inflation and Unemployment

Let us assume that the rate of inflation is negatively related to the level of unemployment and positively to the expected rate of inflation in a dependence known as the Phillips relation such that

\[ \dot{p} = \alpha - \beta U + h\pi^4 \quad \alpha, \beta > 0 \quad 0 < h \leq 1 \]

4 The original idea underlying the model was expressed by A. W. Phillips in a path-breaking paper titled “The Relationship between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957,” *Economica*, November 1958, pp. 283-299. The expanded version of the Phillips relation incorporates the growth rate of money wage \( \dot{w} \) where the rate of inflation is the difference between the increase in wage and the increase in labor productivity \( \dot{T} \), that is, \( \dot{p} = \dot{w} - \dot{T}. \) Thus, inflation would result only when wage increases faster than productivity. Furthermore, wage growth is negatively related to
where \( \dot{p} = \frac{p'}{p} \) is the rate of growth of the price level (that is, the inflation rate), \( U \) is the rate of unemployment and \( \pi \) denotes the expected rate of inflation. Thus, the expectation of higher inflation shapes the behavior of firms and individuals in a way that stimulates inflation. Expecting prices to rise, they might decide to buy more immediately. As people expect inflation to go down (as a result of appropriate government policies, for example) this brings actual inflation down. This version of the Phillips relation that accounts for the expected rate of inflation is called the *expectations-augmented* Phillips relation. The *adaptive expectations* hypothesis further shows how inflationary expectations are formed. The equation

\[
\frac{d\pi}{dt} = j(\dot{p} - \pi) \quad 0 < j \leq 1
\]

illustrates that when the actual rate of inflation exceeds the expected one, this nurtures people’s expectations, so \( \frac{d\pi}{dt} > 0 \). In the opposite case, if the actual inflation is below the expected one, this makes people believe that inflation would go down, so \( \pi \) is reduced. If the projected and the real inflation turn out to be equal, people do not expect a change in the level of inflation.

There is also the reverse effect – that of inflation on unemployment. Thus, when inflation is high for too long, for example, this may discourage people from saving, consequently reducing aggregate investment and increasing the rate of unemployment. We can write that

\[
\frac{dU}{dt} = -k(m - \dot{p}) \quad k > 0
\]

or unemployment increases proportionally with real money where \( \dot{m} \) is the rate of growth of nominal money. Thus, the expression \( (m - \dot{p}) \) gives the rate of growth of real money, or the difference between the growth rate of nominal money and the rate of inflation

\[
m - \dot{p} = \frac{m'}{m} - \frac{p'}{p} = r_{m/p}
\]

where real money is nominal money divided by the average price level in the economy. The model then becomes

\[
\dot{p} = \alpha - \beta U + h\pi \quad \alpha, \beta > 0 \quad 0 < h \leq 1 \quad \text{(expectations-augmented Philips relation)}
\]

\[
\frac{d\pi}{dt} = j(\dot{p} - \pi) \quad 0 < j \leq 1 \quad \text{(adaptive expectations)}
\]

\[
\frac{dU}{dt} = -k(m - \dot{p}) \quad k > 0 \quad \text{(monetary policy)}
\]

We can substitute the first equation into the second, which gives

\[
\frac{d\pi}{dt} = j(\alpha - \beta U + h\pi - \pi)
\]

\[
\frac{d\pi}{dt} = j(\alpha - \beta U) + j(h-1)\pi
\]

Differentiating further with respect to time \( t \),

unemployment and positively to the expected rate of inflation or \( \dot{w} = -\beta U + h\pi \) where \( U \) is the rate of unemployment and \( \pi \) is the expected rate of inflation. If inflationary trends persist long enough, people start forming further inflationary expectations that shape their money-wage demands.
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\[ \frac{d^2 \pi}{dt^2} = -j \beta \frac{dU}{dt} + j(h-1) \frac{d\pi}{dt} \]

and substituting for \( \frac{dU}{dt} \), we obtain

\[ \frac{d^2 \pi}{dt^2} = j \beta k (m - \dot{p}) + j(h-1) \frac{d\pi}{dt} \]

where the second equation of the model implies \( \dot{p} = \frac{1}{j} \frac{d\pi}{dt} + \pi \). Substituting this last expression for \( \dot{p} \), we get

\[ \frac{d^2 \pi}{dt^2} = j \beta k \left( m - \frac{1}{j} \frac{d\pi}{dt} - \pi \right) + j(h-1) \frac{d\pi}{dt} \]

which is a second-order differential equation in \( \pi \). Transforming the equation,

\[ \frac{d^2 \pi}{dt^2} = j \beta km - \beta k \frac{d\pi}{dt} - j \beta k \pi + j(h-1) \frac{d\pi}{dt} \]

\[ \frac{d^2 \pi}{dt^2} + \left( \beta k + j(1-h) \right) \frac{d\pi}{dt} + j \beta k \pi = j \beta km \]

or more simply

\[ \pi'' + \left( \beta k + j(1-h) \right) \pi' + j \beta k \pi = j \beta km \]

where, given the properties of second-order differential equations, we have

\[ a_1 = \beta k + j(1-h) \quad a_2 = j \beta k \quad b = j \beta km \]

The coefficients \( a_1 \) and \( a_2 \) are both positive in view of the signs of the parameters. We can immediately find the equilibrium rate of expected inflation to be the particular integral

\[ \pi_p = \frac{b}{a_2} = m \]

Thus, the intertemporal equilibrium of the expected rate of inflation is exactly the rate of growth of nominal money. To find the time path of \( \pi \) we need to find the characteristic roots of the differential equation by the usual formula

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \]

The time path of \( \pi \) would depend on the particular values of the parameters. Once we find this time path, we might be able to determine that of unemployment \( U \) or the rate of inflation \( \dot{p} \).

Inflation and Unemployment: An Extended Model

Following Olivier Blanchard’s book *Macroeconomics*, we can assume that the rate of change of the inflation rate \( \dot{p} \) is proportional to the difference between the actual unemployment rate \( U \) and the natural rate of unemployment \( U_n \) such that

\[ \frac{dp}{dt} = -\alpha(U - U_n) \quad \alpha > 0 \]

---

Thus, when \( U > U_n \), that is, the actual rate of unemployment exceeds the natural rate, the inflation rate decreases and when \( U < U_n \), the inflation rate increases. The intuitive logic behind this is that in bad economic times when many people are laid off, prices tend to fall. At this point, the actual unemployment would exceed the normal levels. In times of a boom in the business cycle, the rate of actual unemployment would be rather low, but high aggregate demand would push prices up. We also assume that \( U_n \) is constant and that at any given time the actual unemployment rate \( U \) is determined by aggregate demand which, on its own, depends on the real value of money supply given by nominal money supply \( M \) divided by the average price level \( p \). Thus, unemployment is negatively related to real money supply \( \frac{M}{p} \) according to the relationship

\[
U = \gamma - \beta \ln \left( \frac{M}{p} \right)
\]

\( \beta, \gamma > 0 \)

By differentiating the first equation with respect to \( t \),

\[
\frac{d^2 p}{d t^2} = -\alpha \frac{dU}{dt}
\]

and the second equation to obtain \( \frac{dU}{dt} \)

\[
\frac{dU}{dt} = -\beta \frac{d}{dt} \ln \left( \frac{M}{p} \right) = -\beta \left( \frac{d \ln M}{dt} - \frac{d \ln p}{dt} \right) = -\beta (\dot{m} - \dot{p})
\]

where we assume that the growth rate of nominal money supply \( \dot{m} \) is constant. This could be in accordance with government planning or systematic monetary policy. Combining the two results yields

\[
\frac{d^2 p}{d t^2} = -\alpha \frac{dU}{dt} = \alpha \beta (\dot{m} - \dot{p})
\]

\[
\frac{d^2 p}{d t^2} + \alpha \beta \dot{p} = \alpha \beta \dot{m}
\]

which is a second-order differential equation in inflation rate \( \dot{p} \). Solving the differential equation, we have \( a_1 = 0 \), \( a_2 = \alpha \beta \) and \( b = \alpha \beta \dot{m} \). Hence, the particular integral is \( \dot{p}_c = \dot{m} \) and the characteristic equation is

\[
r^2 + \alpha \beta = 0
\]

\[
r_{1,2} = \pm \sqrt{\alpha \beta} i \quad \text{where} \quad m = 0 \quad \text{and} \quad n = \sqrt{\alpha \beta}
\]

Thus, the general solution takes the form

\[
\dot{p}(t) = \dot{m} + e^{0} \left( B_1 \cos \sqrt{\alpha \beta} t + B_2 \sin \sqrt{\alpha \beta} t \right) = \dot{m} + B_1 \cos \sqrt{\alpha \beta} t + B_2 \sin \sqrt{\alpha \beta} t
\]

Since the real part is zero, the function of inflation rate displays regular oscillations about the rate of growth of money supply, which gives the equilibrium level of inflation. To find the time path of unemployment \( U \), we express \( \frac{d\dot{p}}{dt} \):

\[
\frac{d\dot{p}}{dt} = \sqrt{\alpha \beta} \left( -B_1 \sin \sqrt{\alpha \beta} t + B_2 \cos \sqrt{\alpha \beta} t \right)
\]

and substitute it into
where the constants $B_1$ and $B_2$ have not been definitized. It follows that the unemployment rate also displays regular oscillations, similar to the inflation rate, but its equilibrium is the natural rate of unemployment. Since the real part is zero, again the time path is neither convergent nor divergent.

**Higher-order Differential Equations**

Recall the $n^{th}$-order differential equation with constant coefficients

$$y^{(n)}(t) + a_1y^{(n-1)}(t) + \ldots + a_{n-1}y'(t) + a_n y = b$$

To solve such higher-order differential equations, we simply repeat the steps applicable to second-order differential equations. With a simplest constant function assumed, such as $y = c$, all derivatives are zero. The particular integral is

$$y_p = c = \frac{b}{a_n} \quad a_n \neq 0$$

If $a_n = 0$, we try $y = ct$ such that $y'(t) = c$; but all other derivatives are zero, so the equation becomes $a_{n-1}c = b$. The particular integral is

$$y_p = ct = \frac{b}{a_{n-1}}t \quad a_n = 0 \quad a_{n-1} \neq 0$$

In the case when $a_n = a_{n-1} = 0$, the solution must be of the type $y = ct^2$. This produces the derivatives $y'(t) = 2ct$ and $y''(t) = 2c$ and the particular integral

$$y_p = ct^2 = \frac{b}{a_{n-2}}t^2 \quad a_n = a_{n-1} = 0 \quad a_{n-2} \neq 0$$

The complementary function is the general solution of the homogeneous equation

$$y^{(n)}(t) + a_1y^{(n-1)}(t) + \ldots + a_{n-1}y'(t) + a_n y = 0$$

If the solution is in the form $y = Ae^{rt}$, the derivatives can be written off as $y'(t) = rAe^{rt}$, $y''(t) = r^2Ae^{rt}$, ..., $y^{(n)}(t) = r^ne^{rt}$. This gives rise to the $n^{th}$-degree characteristic equation with $n$ roots

$$Ae^{rt}(r^n + a_1r^{n-1} + \ldots + a_{n-1}r + a_n) = 0 \quad \text{or}$$

$$r^n + a_1r^{n-1} + \ldots + a_{n-1}r + a_n = 0$$

Thus, the complementary function, with all roots real and distinct, can be written as

$$y_c = A_1e^{r_1t} + A_2e^{r_2t} + \ldots + A_ne^{r_nt}$$

Assuming that the first two roots are repeated, we have $r_1 = r_2$, so the first two terms of the general solution to the differential equation can be written as $A_1e^{r_1t} + A_2te^{r_1t}$. If we assume further that the next two roots are complex such that $r_3, 4 = m \pm ni$, the general solution of this differential equation can be written as

$$y(t) = A_1e^{r_1t} + A_2te^{r_1t} + e^{rt}(B_1\cos nt + B_2\sin nt) + y_p$$
To find the values of the four arbitrary constants, we need four initial conditions.

**Example:** Solve the differential equation \( y''(t) + 3y'(t) + 5y(t) = 15 \). Since the highest derivative is the third-order derivative \( y'''(t) \), this is a third-order differential equation with a particular integral \( y_p = \frac{-15}{5} = -3 \). Its characteristic equation is \( r^3 + r^2 + 3r - 5 = 0 \), which can be factored out into

\[(r - 1)(r^2 + 2r + 5) = 0\]

with one real root \( r_1 = 1 \) and a pair of complex conjugate roots \( r_{1,2} = -1 \pm 2j \). The general solution, therefore, is

\[y(t) = A_1e^t + e^{-t}(B_1 \cos 2t + B_2 \sin 2t) - 3\]

**Second-order Difference Equations**

We recall that first-order difference equations involve terms like \( y_{t+1} \) and \( y_t \), where the difference in each period is given. Thus knowing some initial value \( y_0 \) we can determine the time path of the \( y \) function as the time factor \( t \) changes. A simple second-order difference equation is

\[y_{t+2} + b_1y_{t+1} + b_2y_t = c\]

To find the particular integral in the simplest case, we can take a solution of the form \( y_t = k \) where in every period \( y \) is the constant \( k \):

\[k + b_1k + b_2k = c\]

\[y_p = k = \frac{c}{1 + b_1 + b_2} \quad \text{where} \quad b_1 + b_2 \neq -1\]

If \( b_1 + b_2 = -1 \), we must adopt \( y_t = kt \). This will result in the terms \( y_{t+1} = k(t+1) \) and \( y_{t+2} = k(t+2) \) for the next time periods. Substituting these expressions into the difference equation gives

\[k = \frac{c}{(1 + b_1 + b_2)t + b_1 + 2} = \frac{c}{b_1 + 2} \quad \text{where} \quad b_1 + b_2 = -1\]

Hence, the particular integral is

\[y_p = kt = \frac{c}{b_1 + 2}t \quad \text{where} \quad b_1 + b_2 = -1 \text{ and } b_1 \neq -2\]

Since the particular integral in this second case involves \( t \), it represents a moving equilibrium. If \( b_1 + b_2 = -1 \) (so that \( b_1 = -2 \) and \( b_2 = 1 \)), then we try a solution of the type \( y_t = kt^2 \) so the other two terms are \( y_{t+1} = k(t+1)^2 \) and \( y_{t+2} = k(t+2)^2 \). They generate \( k = \frac{c}{2} \) and

\[y_p = kt^2 = \frac{c}{2}t^2 \quad \text{where} \quad b_1 = -2 \text{ and } b_2 = 1\]

**The Complementary Function**

With first-order difference equations, we found that the expression \( y_t = Aa^t \) describes well the general solution of such an equation, and we try it to find the complementary function. This implies that \( y_{t+1} = Aa^{t+1} \) and \( y_{t+2} = Aa^{t+2} \), which upon substitution in
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\[ y_{t+2} + b_1 y_{t+1} + b_2 y_t = 0 \] yields

\[ Aa^{t+2} + b_1 Aa^{t+1} + b_2 Aa^t = 0 \quad \text{or} \]
\[ a^2 + b_1 a + b_2 = 0 \]

This characteristic equation has the roots \( a_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} \). Hence, for the complementary function, we have three possibilities again:

**Case 1. Distinct real roots**

If \( b_1^2 > 4b_2 \), then both roots are real and different, so the complementary function is

\[ y_c = y_1 + y_2 = A_1 a_1^t + A_2 a_2^t \]

**Example:** Solve the second-order difference equation \( y_{t+2} + 3y_{t+1} - 4y_t = 15 \). Here we have \( b_1 = 3 \), \( b_2 = -4 \), and \( c = 12 \). We also note that \( b_1 + b_2 = -1 \), so

\[ y_p = \frac{c}{b_1 + 2} = \frac{15}{3 + 2} t = 3t \]

The characteristic equation is \( a^2 + 3a - 4 = 0 \) with roots \( a_1 = 1 \) and \( a_2 = -4 \). Thus, the general solution is

\[ y_i = y_c + y_p = A_1 + A_2 (-4)^t + 3t \]

Suppose we are also given that \( y_o = 3 \) and \( y_1 = 1 \) for the two periods \( t = 0 \) and \( t = 1 \), respectively. Substituting these values for \( t \), we obtain

\[ y_o = A_1 + A_2 = 3 \]
\[ y_1 = A_1 - 4A_2 + 3 = 1 \]

Thus, the constants are \( A_1 = 2 \) and \( A_2 = 1 \) and the final solution is

\[ y_t = 2 + (-4)^t + 3t \]

**Case 2. Single real root**

If \( b_1^2 = 4b_2 \), there is only one real root \( a = \frac{-b_1}{2} \). Then the complementary function is

\[ y_c = y_1 + y_2 = A_1 a^t + A_2 ta^t \]

**Example:** Solve the second-order difference equation \( y_{t+2} - 2y_{t+1} + y_t = 8 \). Here we have \( b_1 = -2 \), \( b_2 = 1 \), and \( c = 8 \). Also, \( b_1 + b_2 = -1 \) implies a particular integral of the type

\[ y_p = \frac{c}{2} t^2 = \frac{8}{2} t^2 = 4t^2 \]

The characteristic equation is \( a^2 - 2a + 1 = 0 \) with a single root \( a = 1 \). The general solution is, therefore,

\[ y_t = y_c + y_p = A_1 + A_2 t + 4t^2 \]

**Case 3. Complex roots**

When \( b_1^2 < 4b_2 \), again a pair of conjugate complex numbers \( a_{1,2} = m \pm ni \) obtains where
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\[ m = \frac{-b_1}{2} \quad \text{and} \quad n = \sqrt{\frac{4b_2 - b_1^2}{2}} \]

The complementary function is

\[ y_c = A_1a_1^t + A_2a_2^t = A_1(m + ni)^t + A_2(m - ni)^t \]

From the De Moivre’s theorem it follows that \((m \pm ni)^t = R^t(\cos \theta t \pm i \sin \theta t)\) where

\[ R = \sqrt{m^2 + n^2} = \sqrt{\frac{b_1^2 + 4b_2 - b_1^2}{4}} = \sqrt{b_2} \]

Here \(\theta\) is measured in radians and \(\cos \theta = \frac{m}{R} = \frac{-b_1}{2\sqrt{b_2}}\) and \(\sin \theta = \frac{n}{R} = \sqrt{1 - \frac{b_1^2}{4b_2}}\). Hence, the complementary function is

\[ y_c = A_1R^t(\cos \theta t + i \sin \theta t) + A_2R^t(\cos \theta t - i \sin \theta t) = R^t(B_1 \cos \theta t + B_2 \sin \theta t) \]

where the multiplicative factor \(R^t\) substitutes the natural exponential term \(e^{mt}\) used in differential equations.

**Example:** Find the general solution to the equation \(y_{t+2} - 3y_{t+1} + 9y_t = 14\). Here we have \(b_1 = -3\), \(b_2 = 9\), and \(c = 14\). For the particular integral,

\[ y_p = \frac{c}{1 + b_1 + b_2} = \frac{14}{1 - 3 + 9} = 2 \]

This is the case of complex roots since \(b_1^2 < 4b_2\), therefore, \(R = \sqrt{b_2} = \sqrt{9} = 3\).

\[ \cos \theta = -\frac{b_1}{2\sqrt{b_2}} = -\frac{3}{2(3)} = \frac{1}{2} \quad \text{and} \quad \sin \theta = \sqrt{1 - \frac{b_1^2}{4b_2}} = \sqrt{1 - \frac{9}{4(9)}} = \frac{\sqrt{3}}{2} \]

From the respective values of the two trigonometric functions, we infer that \(\theta = \frac{\pi}{3}\). Hence, the general solution is

\[ y_t = y_c + y_p = 3^t \left( B_1 \cos \frac{\pi}{3} t + B_2 \sin \frac{\pi}{3} t \right) + 2 \]

**Dynamic Stability**

For the second-order difference equation

\[ y_{t+2} + b_1y_{t+1} + b_2y_t = c \]

we have the term \(a^t\), which may show oscillatory behavior depending on the value of the base \(a\). In the case of two distinct roots, if \(|a_1| > 1\) and \(|a_2| > 1\), both terms in the complementary function \(y_c = A_1a_1^t + A_2a_2^t\) will be explosive and the time path of \(y_t\) is divergent (the time path is oscillatory, if \(a < 0\)). When \(|a_1| < 1\) and \(|a_2| < 1\), both terms will converge to zero as \(t \to \infty\) and the time path is convergent. If either \(|a_1|\) or \(|a_2|\) is greater than 1, the time path is divergent. With a single root, the function is dynamically stable if \(|a| < 1\). For the case of complex roots, we found the solution to be

\[ y_t = R^t(B_1 \cos \theta t + B_2 \sin \theta t) + y_p \]
The parentheses expression shows a fluctuating path because it contains circular functions. The fluctuation would be a stepped (nonsmooth) fluctuation, rather than oscillation. If \( R < 1 \), the time path would be dynamically stable. Since \( R \) is the absolute value of the conjugate complex roots \( m \pm ni \), the condition for convergence is again that the characteristic roots be less than 1. In all cases, the time path of \( y_t \) will be dynamically stable if the absolute value of every root is less than 1.

The Multiplier-Accelerator Model

The multiplier-accelerator model shows the interaction between aggregate investment and output. Usually, in the presence of positive exogenous shocks, increased investment has a multiplying effect on GDP by the amount of the investment multiplier, but the increase in GDP makes firms believe that demand for their goods has increased. This stimulates them to invest more in capital stock, a process known as the accelerator. Thus, investment stimulates GDP through the multiplier process while GDP further pushes up investment through the accelerator process in an interactive way. Of course, a downturn in the economy would have an effect opposite to the multiplier-accelerator process or would force the economy to contract. The model was first advanced by Paul Samuelson, who extended the Keynesian national-income model of the investment multiplier by the accelerator principle.\(^6\) The model assumes the following three equations:

\[
Y_t = C_t + I_t + G_o
\]

\[
C_t = \beta Y_{t-1} \quad \text{with} \quad 0 < \beta < 1
\]

\[
I_t = \alpha (C_t - C_{t-1}) \quad \text{with} \quad \alpha > 0
\]

People spend based on income earned in the previous period where \( \beta \) shows the share of income that is consumed, that is, the marginal propensity to consume. Furthermore, investment is positively related to the increase in aggregate consumption \( \Delta C_{t-1} = \alpha (C_t - C_{t-1}) \) showing here the accelerator effect. That is, based on increased consumption, firms expect demand for their product to rise and, hence, decide to increase investment. Note also that the parameter \( \alpha \) is called an accelerator coefficient and is greater than zero. Substituting the respective terms for \( C_t \) in the last equation, we obtain

\[
I_t = \alpha (\beta Y_{t-1} - \beta Y_{t-2})
\]

and substitute this new result and the second equation into the first one:

\[
Y_t = \beta Y_{t-1} + \alpha \beta Y_{t-2} - \alpha \beta Y_{t-2} + G_o
\]

\[
Y_{t+2} - \beta (1 + \alpha) Y_{t+1} + \alpha \beta Y_t = G_o
\]

The parameters in this second-order difference equation are \( b_1 = -\beta (1 + \alpha) \), \( b_2 = \alpha \beta \), and \( c = G_o \). We can easily find the particular integral as

\[
Y_p = \frac{c}{1 + b_1 + b_2} = \frac{G_o}{1 - \beta (1 + \alpha) + \alpha \beta} = \frac{G_o}{1 - \beta}
\]

Since \( \beta \) is less than 1, we could expect a meaningful intertemporal equilibrium for national income that is positively related to exogenous government spending. Furthermore, \( \frac{1}{1 - \beta} \) is the value of the multiplier. The characteristic equation of the model is

\[
a^2 - \beta (1 + \alpha) a + \alpha \beta = 0 \quad \text{where} \quad a_{1,2} = \frac{\beta (1 + \alpha) \pm \sqrt{\beta^2 (1 + \alpha)^2 - 4 \alpha \beta}}{2}
\]

---

where we know the two roots satisfy the conditions

\[ a_1 + a_2 = \beta(1 + \alpha) \quad \text{and} \quad a_1a_2 = \alpha\beta \]

Thus,

\[ (1-a_1)(1-a_2) = 1 - (a_1 + a_2) + a_1a_2 = 1 - \beta(1+\alpha) + \alpha\beta = 1 - \beta \]

which implies that \( 0 < (1-a_1)(1-a_2) < 1 \). Furthermore, for the complementary function, there are three possible cases depending on whether \( b_1^2 > 4b_2 \) or not. This first case is equivalent to

\[ \beta^2(1+\alpha)^2 > 4\alpha\beta \]

\[ \beta > \frac{4\alpha}{(1+\alpha)^2} \]

In this first case of distinct real roots since \( a_1a_2 > 0 \) and \( a_1 + a_2 > 0 \), both roots are positive. This precludes oscillation, and convergence would depend on whether \( a_1 \) and \( a_2 \) are smaller or bigger than 1. Several cases might be considered, but the legitimate ones are presented in Table 2. Similar is the case of a single real root \( a = \frac{\beta(1+\alpha)}{2} \), which is positive again. Oscillation is excluded and the dynamic stability of national income depends on whether \( a \) is smaller or bigger than 1. In the case of conjugate complex roots the presence of \( R = \sqrt{b_1} = \sqrt{\alpha\beta} \) determines stepped fluctuation. If \( R < 1 \), the fluctuation would be narrowed down, while for \( R \geq 1 \) we would have explosive growth. These conclusions are summarized in the table below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Subcase</th>
<th>Time path of ( Y_t )</th>
<th>Dynamic stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Distinct real roots</td>
<td>( \beta &gt; \frac{4\alpha}{(1+\alpha)^2} )</td>
<td>( 0 &lt; a_1 &lt; a_2 &lt; 1 )</td>
<td>Nonoscillatory</td>
</tr>
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</tr>
</tbody>
</table>

Table 2

In conclusion, the time path of national income is convergent only if \( \alpha\beta < 1 \) in all cases. Furthermore, the model shows that it is possible for national income to have cyclical fluctuations endogenously without any external shocks present, but merely due to the interactive play between the multiplier and the accelerator process.
Inflation and Unemployment in Discrete Time

Recall that the model of the augmented Philips curve showing the relationship between inflation and unemployment in continuous time, that is, with differential equations, took the form:

\[
\begin{align*}
\dot{p} &= \alpha - \beta U + h \pi \\
\frac{d\pi}{dt} &= j(\dot{p} - \pi) \\
\frac{dU}{dt} &= -k(\bar{m} - \dot{p})
\end{align*}
\]

We can transform this model in a discrete-time form so it becomes

\[
\begin{align*}
\dot{p}_t &= \alpha - \beta U_t + h \pi_t \\
\pi_{t+1} - \pi_t &= j(\dot{p}_t - \pi_t) \\
U_{t+1} - U_t &= -k(\bar{m} - \dot{p}_{t+1})
\end{align*}
\]

In solving the discrete-time model, we notice the difference terms for expected inflation \( \pi_t \) and unemployment \( U_t \) that obtain in the second and the third equation. To take advantage of these differences, we can further express the difference for actual inflation, which is

\[\Delta p_t = p_{t+1} - p_t\]

where we substitute \( \dot{p}_{t+1} = \alpha - \beta U_{t+1} + h \pi_{t+1} \)

\[
\begin{align*}
\dot{p}_{t+1} - \dot{p}_t &= \alpha - \beta U_{t+1} + h \pi_{t+1} - \alpha + \beta U_t - h \pi_t = -\beta(U_{t+1} - U_t) + h(\pi_{t+1} - \pi_t)
\end{align*}
\]

and substituting the last two equations of the model into this expression,

\[
\begin{align*}
\dot{p}_{t+1} - \dot{p}_t &= \beta k(m - \dot{p}_{t+1}) + hj(\dot{p}_t - \pi_t) \\
(1 + \beta k)\dot{p}_{t+1} - (1 + hj)\dot{p}_t + hj\pi_t &= \beta km
\end{align*}
\]

Since the \( \pi_t \) term appears in the difference equation for price, we can express it from the first equation of the model as \( h \pi_t = \dot{p}_t - \alpha + \beta U_t \) and substitute:

\[
\begin{align*}
(1 + \beta k)\dot{p}_{t+1} - (1 + hj)\dot{p}_t + j\dot{p}_t - j\alpha + j\beta U_t &= \beta km \\
(1 + \beta k)\dot{p}_{t+1} - (1 - j + hj)\dot{p}_t + j\beta U_t &= \beta km + j\alpha
\end{align*}
\]

We still have one term \( U_t \) to get rid of. In order to do that, we can extend the upper equation by one period so that it becomes

\[
\begin{align*}
(1 + \beta k)\dot{p}_{t+2} - (1 - j + hj)\dot{p}_{t+1} + j\beta U_{t+1} &= \beta km + j\alpha
\end{align*}
\]

From the monetary policy equation in the original model, we know that the difference for unemployment is \( U_{t+1} - U_t = -k(m - \dot{p}_{t+1}) \), and we use this when subtracting the last two equations:

\[
(1 + \beta k)\dot{p}_{t+2} - (1 - j + hj + 1 + \beta k)\dot{p}_{t+1} + (1 - j + hj)\dot{p}_t + j\beta(U_{t+1} - U_t) = 0
\]

After substituting for the difference of unemployment and some further transformations, we reach the following second-order difference equation in \( \dot{p} \):

\[
(1 + \beta k)\dot{p}_{t+2} - (1 - j + hj + 1 + \beta k)\dot{p}_{t+1} + (1 - j + hj)\dot{p}_t + j\beta(U_{t+1} - U_t) = 0
\]

Here the parameters are
The particular integral can be immediately found as

\[ p' = \omega = \frac{c \beta}{1 + b_1 + b_2} = \frac{j \beta \kappa n}{1 + \beta k} = \dot{m} \]

The characteristic roots must satisfy the conditions

\[ a_1 + a_2 = -b_1 = \frac{1 + hj}{1 + \beta k} + 1 - j \]

From the values of the respective parameters, we can conclude that

\[ a_1, a_2 > 0 \quad a_1 a_2 \in (0, 1) \quad \text{and} \quad (1 - a_1)(1 - a_2) > 0 \]

Hence, \( a_1 \) and \( a_2 \) both are positive fractions and the time path of inflation \( \dot{p} \) is convergent and nonoscillatory with distinct or repeated roots – that is, when \( b_2^2 \geq 4b_2 \). If \( b_2^2 < 4b_2 \), we have complex roots where \( R = \sqrt{b_2} \). Since \( b_2 \) itself is a positive fraction, so must be \( R \) which renders the time path of \( \dot{p} \) convergent in the form of stepped fluctuation.

We can as well study the behavior of unemployment \( U \) in time. From the last equation of the model,

\[ U_{t+1} - U_t = -k(\dot{m} - \dot{p}_{t+1}) \]

But from the expectations-augmented Philips curve equation, we know that \( \dot{p}_{t+1} = \alpha - \beta U_{t+1} + h\pi_{t+1} \).

Substituting this expression in the difference equation for unemployment gives

\[ U_{t+1} - U_t = -k(\dot{m} - \alpha + \beta U_{t+1} - h\pi_{t+1}) \]

Extending this by one time period,

\[ U_{t+2} - U_{t+1} = -k(\dot{m} - \alpha + \beta U_{t+2} - h\pi_{t+2}) \]

and subtracting the last two equations, we obtain

\[ (1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = k\pi_{t+2} - \pi_{t+1} \]

From the adaptive-expectations equation we know that

\[ \pi_{t+2} - \pi_{t+1} = j(\dot{p}_{t+1} - \pi_{t+1}) \]

So, substituting this difference term for expected inflation results in

\[ (1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = k\pi_{t+2} - \pi_{t+1} \]

where \( \dot{p}_{t+1} = \alpha - \beta U_{t+1} + h\pi_{t+1} \)

\[(1 + \beta k)U_{t+2} - (2 + \beta k)U_{t+1} + U_t = k\pi_{t+2} - \pi_{t+1} \]

or

\[(1 + \beta k)U_{t+2} - (2 + \beta k - \beta kjh)U_{t+1} + U_t = k\pi_{t+2} - \pi_{t+1} \]

From the original difference equation for unemployment, we express the term \( \pi_{t+1} \)

\[ \frac{U_{t+1} - U_t}{k} = -\dot{m} + \alpha - \beta U_{t+1} + h\pi_{t+1} \]

\[ \pi_{t+1} = \frac{(1 + \beta k)U_{t+1} - U_t + km - ak}{hk} \]

Substituting this finally gives a second-order difference equation solely in \( U \).
(1 + \beta k)U_{t+2} - (2 + \beta k - \beta kjh)U_{t+1} + U_t = \frac{khj\alpha + kj(h-1)(1 + \beta k)U_{t+1} - U_t + km - \alpha k}{hk} \) 

(1 + \beta k)U_{t+2} - (2 + \beta k - \beta kjh)U_{t+1} - j(h-1)(1 + \beta k)U_{t+1} + (1 - j + jh)U_t = 
= khj\alpha + jk(h-1)(m - \alpha) 

(1 + \beta k)U_{t+2} - \left[1 + hj + (1 + \beta k)(1 - j)\right]U_{t+1} + (1 - j + jh)U_t = kj\left[\alpha h + (h-1)m - \alpha h + \alpha\right] 
\frac{1 + \beta k}{1 + \beta k}U_{t+1} + \frac{(1 - j + jh)}{1 + \beta k}U_t = \frac{kj\left[\alpha + (h-1)m\right]}{1 + \beta k} 

The parameters here are 

\[ b_1 = \frac{1 + hj + (1 + \beta k)(1 - j)}{1 + \beta k}, \quad b_2 = \frac{(1 - j + jh)}{1 + \beta k} \quad \text{and} \quad c = \frac{kj\left[\alpha + (h-1)m\right]}{1 + \beta k} \]

It can be checked that the intertemporal equilibrium level of unemployment is 

\[ U_p = \frac{c}{1 + b_1 + b_2} = \frac{\alpha + (h-1)m}{\beta} \]

Since in a state of general equilibrium the equilibrium rate of inflation was found to be exactly the growth rate of money \( m \), the last equation could be written as 

\[ \bar{U} = \frac{\alpha + (h-1)m}{\beta} \]

Therefore, the equilibrium inflation and unemployment must be negatively related in the long run, a relationship which we previously denoted as the long-run Phillips curve. In the special case of \( h = 1 \), the \( \dot{p} \) term will drop out of the equation and the unemployment rate will become a constant. This will give rise to a vertical long-run Phillips curve with the unemployment rate plotted on the horizontal axis. This fixed value of unemployment which represents the natural rate of unemployment teaches economic policy-makers that inflation and unemployment may be unrelated in the long run. If \( h < 1 \), the coefficient of \( \dot{p} \) will be negative and the long-run Phillips curve will be negatively sloped. Thus, the value of the \( h \) parameter determines the trade-off between inflation and unemployment. By definition this parameter measures the degree to which expectations form actual inflation, that is, the interrelationship between expected and actual inflation. Thus, as stronger expectations of higher inflation form in the nation and penetrate the wage structure of the economy, there will be little interdependence between inflation and unemployment and little the government can do. With lower inflationary expectations, the potential for government policies to take advantage of the trade-off between inflation and unemployment increases.

**Higher-order Difference Equations**

To find the particular integral of the \( n^{th} \) order difference equation with constant coefficients and a constant term, 

\[ y_{t+n} + b_1y_{t+n-1} + \ldots + b_{n-1}y_{t+1} + b_ny_t = c \]

we again try solutions \( y_t = k, \quad y_t = kt, \quad y_t = kt^2 \), etc. As to the complementary equation, an \( n \)th-degree characteristic equation obtains 

\[ a^n + b_1a^{n-1} + \ldots + b_{n-1}a + b_n = 0 \]

with \( n \) characteristic roots \( a_i \ (i = 1, 2, \ldots, n) \). If some of the roots are repeated (for instance, the first two), and the next two are complex numbers, the general solution would be
$y_t = A_1a_t^3 + A_2ta_t^2 + R'(B_1\cos \theta t + B_2\sin \theta t) + y_p$

At least $n$ initial conditions are necessary to find the values of the $n$ arbitrary constants.

**Example:** Find the general solution of the third-order difference equation

$$y_{r+3} - \frac{5}{4}y_{r+2} + \frac{1}{2}y_{r+1} - \frac{1}{16}y_r = 3$$

Trying a solution of the type $y_r = k$, we find the particular integral to be

$$y_p = \frac{c}{1 + b_1 + b_2 + b_3} = \frac{3}{1 - \frac{5}{4} + \frac{1}{2} - \frac{1}{16}} = \frac{3(16)}{16 - 20 + 8 - 1} = 16$$

The characteristic equation is

$$a^3 - \frac{5}{4}a^2 + \frac{1}{2}a - \frac{1}{16} = 0$$

Factoring out the term $a - \frac{1}{4}$, we transform this cubic equation into

$$\left(a - \frac{1}{4}\right)^2 = 0$$

with roots $a_1 = a_2 = \frac{1}{2}$ and $a_3 = \frac{1}{4}$. Hence,

$$y_t = A_1\left(\frac{1}{2}\right)^t + A_2t\left(\frac{1}{2}\right)^t + A_3\left(\frac{1}{4}\right)^t + 16$$

Since $a_1, a_2, a_3 < 1$, $y_t$ converges to the stationary intertemporal equilibrium of 16.

**Problems**

1. The logistic model of population growth (also known as the Verhulst model) assumes that the growth rate of a population decreases as this population grows in size. Similar to Malthusian growth, it assumes that there are limits to the increase of human population. This might be due to a decline in the arable or other land as a fixed input, the depletion of nonrenewable resources, crowding, and the eventual spreading of epidemics that could reduce the human population considerably. Thus, if $y$ is the cumulative world population and its growth rate is $\dot{y}$, according to the model

$$\dot{y} = a - by$$

$a, b > 0$

Find the time path of human population using the logistic model.

**Solution:**

We can rewrite the equation as

$$\frac{1}{y} \frac{dy}{dt} = a - by$$

or

$$\left(\frac{1}{y} + \frac{a}{b - ay}\right)dy = bdt$$

Integrating both sides of the equation,
\[
\ln \left| \frac{y}{b-ay} \right| = bt + c
\]

and taking the antilog of both sides,

\[
\left| \frac{y}{b-ay} \right| = e^{bt} \text{ or } \left| \frac{y}{b-ay} \right| = Ae^{bt}
\]

which transforms into \( y = \frac{Abe^{bt}}{1 + Ae^{bt}} \). To definitize the \( A \) constant, we set \( t = 0 \)

\[
y(0) = \frac{Ab}{1 + Aa}
\]

which gives \( A = \frac{y(0)}{b - ay(0)} \)

Substituting \( A \) and transforming further gives the definite solution of population

\[
y(t) = \frac{by(0)e^{bt}}{b - ay(0) + ay(0)e^{bt}}
\]

2. The population of a country grows according to the logistic equation \( \dot{y} = a - by \) where the rate of change of the population with time is \( \frac{dy}{dt} \) and \( a, b > 0 \). Find the equilibrium size of this population – that is, the one for which the rate of change is zero.

Solution:

In order for the population to be in equilibrium we need to have \( y'(t) = 0 \). From the solution obtained previously, we have \( y(t) = \frac{by(0)e^{bt}}{b - ay(0) + ay(0)e^{bt}} \), so differentiating with respect to \( t \),

\[
y'(t) = \frac{b^2 y(0)e^{bt} \left[ b - ay(0) + ay(0)e^{bt} \right] - ab^2 \left[ y(0) \right]^2 e^{2bt}}{\left[ b - ay(0) + ay(0)e^{bt} \right]^2} = \frac{b^2 y(0)e^{bt} \left[ b - ay(0) \right]}{\left[ b - ay(0) + ay(0)e^{bt} \right]^2} = 0
\]

which implies that the expression \( b - ay(0) \) in the numerator should be 0. Thus, we obtain \( y(0) = \frac{b}{a} \) for the initial condition in equilibrium. Substituting to find that equilibrium value,

\[
y(t) = \frac{b e^{bt}}{a(b - b + be^{bt})} = \frac{b}{a}
\]

Thus, we have found that the equilibrium population size is \( b/a \). Depending on whether the initial population is less than or greater than \( b/a \), there will be growth or decline in the population according to the equation of logistic growth.
3. For the logistic growth of the population discussed in the previous problem, prove that the maximum rate of growth occurs when the population is equal to half its equilibrium size, that is, when the population is \( b/2a \).

**Solution:**

In order for the growth rate to be maximum we need to find the maximum of \( dy/dt \). This means that we can express \( \frac{d^2y}{dt^2} \) and set it equal to zero. From the logistic equation, we have

\[
\frac{dy}{dt} = y(b - ay)
\]

or given the solution for \( y = \frac{by(0)e^{bt}}{b - ay(0) + ay(0)e^{bt}} \)

\[
y'(t) = \frac{by(0)e^{bt}}{b - ay(0) + ay(0)e^{bt}} \left[ b - ay(0) + ay(0)e^{bt} \right] = \frac{by(0)e^{bt} \left[ b^2 - aby(0) \right]}{\left[ b - ay(0) + ay(0)e^{bt} \right]^2}
\]

Differentiating further to find \( y''(t) \),

\[
y''(t) = \frac{by(0)\left[ b^2 - aby(0) \right]be^{bt} \left[ b - ay(0) + ay(0)e^{bt} - 2ay(0)e^{bt} \right]}{\left[ b - ay(0) + ay(0)e^{bt} \right]^3} = 0
\]

which implies that the numerator would be zero when

\[
b - ay(0) = 0 \quad \text{or} \quad b - ay(0) - ay(0)e^{bt} = 0
\]

But the first case implies that \( y' = 0 \), which cannot give the condition for maximum growth. We have also discussed this case. Thus, we review the second, which gives

\[
y(0) = \frac{b}{a(1 + e^{bt})}
\]

Substituting the initial value in the expression for \( y(t) \),

\[
y(t) = \frac{b^2 e^{bt}}{a(1 + e^{bt}) \left( b - \frac{b}{1 + e^{bt}} + \frac{be^{bt}}{1 + e^{bt}} \right)} = \frac{b^2 e^{bt}}{a2be^{bt}} = \frac{b}{2a}
\]

which proves that the maximum growth rate of the population is achieved when population is \( b/2a \).

4. Imagine that the Isle of Timbuktu has an initial population of 100,000 and an equilibrium population of 1 million. The population is known to have a logistic growth pattern. Statisticians count that at the end of one year the population doubles from the initial moment, that is, there are 200,000 citizens of Timbuktu. Determine the time path of Timbuktu’s population. What is the time at which the population is increasing most rapidly?
Solution:

From the general result for the logistic equation of the type \( \frac{dy}{dt} = y(b - ay) \), the time path of the function is given by \( y(t) = \frac{by(0)e^{bt}}{b - ay(0) + ay(0)e^{bt}} \). Here we have

\[
y(0) = 100,000 \quad \bar{y} = \frac{b}{a} = 1,000,000 \quad y(1) = 200,000
\]

\[
y(t) = \frac{by(0)e^{bt}}{a \left[ \frac{b}{a} - y(0) + y(0)e^{bt} \right]}
\]

Substituting in the equation,

\[
y(t) = \frac{1,000,000(100,000)e^{bt}}{(1,000,000 - 100,000 + 100,000e^{bt})} = \frac{10^6 e^{bt}}{9 + e^{bt}}
\]

To find \( b \), we use that \( y(1) = 200,000 \)

\[
y(1) = \frac{10^6 e^b}{9 + e^b} = 200,000 \quad \text{which transforms into}
\]

\[
5e^b = 9 + e^b
\]

\[
4e^b = 9
\]

\[
e^b = 2.25
\]

\[
b = \ln 2.25
\]

Hence, the time path for the Timbuktu population is

\[
y(t) = \frac{10^6 e^{b\ln 2.25}}{9 + e^{b\ln 2.25}}
\]

Maximum growth would be achieved when \( y = \frac{b}{2a} \) or \( y = 500,000 \). Substituting in the obtained function,

\[
500,000 = \frac{10^6 e^{b\ln 2.25}}{9 + e^{b\ln 2.25}}
\]

\[
1 = \frac{2e^{b\ln 2.25}}{9 + e^{b\ln 2.25}}
\]

\[
9 + e^{b\ln 2.25} = 2e^{b\ln 2.25}
\]

\[
e^{b\ln 2.25} = 9
\]

Taking the log of both sides,

\[
t \ln 2.25 = \ln 9
\]

\[
t = \frac{\ln 9}{\ln 9 - \ln 4} \approx 2.7 \text{ years}
\]

The population will grow most rapidly after 2.7 years.

5. A textile factory has 300 workers, all of whom are vulnerable to the Brisbane flu virus. An epidemic is known to have spread out where the number of infected workers is \( I \). The rate of change with respect to time of the number of infected workers is proportional both to the number of infected and the number of uninfected, that is, \( 300 - I \), according to the equation
where $k$ is a constant of proportionality. Find the number of infected people at time $t$ days, if at time $t = 0$ one worker becomes infected. If $k = 0.01$, find the value of the rate of new cases $I'(t)$ after 3 days, that is, $I'(3)$.

**Solution:**

The equation can be rewritten as

$$\frac{dI}{dt} = I(300k - kI)$$

which implies $b = 300k$ and $a = k$. We also know that $I(0) = 1$. Substituting in the general solution,

$$I(t) = \frac{bI(0)e^{bt}}{b - aI(0) + aI(0)e^{at}}$$

$$I(t) = \frac{300ke^{300kt}}{300k - k + ke^{300kt}} = \frac{300e^{300kt}}{299 + e^{300kt}}$$

Setting $k = 0.01$, we have

$$I(t) = \frac{300e^{3t}}{299 + e^{3t}}$$

Differentiating to find $I'(t)$,

$$I'(t) = \frac{300\left[3e^{3t}(299 + e^{3t}) - 3e^{6t}\right]}{(299 + e^{3t})^2} = \frac{900(299)e^{3t}}{(299 + e^{3t})^2} = \frac{269,100e^{3t}}{(299 + e^{3t})^2}$$

After 3 days, the rate of new cases is

$$I'(t) = \frac{269,100e^9}{(299 + e^9)^2}$$

6. The growth rate of a certain population depends on the supply of food which changes seasonally. The growth of the population is given by the equation

$$\frac{dp}{dt} = cp(t) \cos t$$

where $c$ is a positive constant. Solve this simple model of seasonal population growth in terms of an initial population $p(0)$. Analyze the time path of the population function.

**Solution:**

We can rearrange the equation in the form

$$\frac{1}{p} \frac{dp}{dt} = c \cos t$$

Integrating both sides with respect to $t$,
\[
\int \frac{1}{p} \frac{dp}{dt} = \int c \cos dt
\]

In \( p = c \sin t + c_1 \) since \( \frac{d \sin t}{dt} = \cos t \)

Taking the antilog of both sides,
\[
p(t) = e^{c \sin t} e^{c_1} = Ae^{c \sin t}
\]

To definitize \( A \), we set \( t = 0 \):
\[
p(0) = Ae^{c \sin 0} = Ae^0 = A \quad \text{where} \quad \sin 0 = 0
\]

which gives the definite solution for population:
\[
p(t) = p(0)e^{c \sin t}
\]

Since \( \sin t \) is a circular function, it fluctuates around -1 and 1. Hence, the population function would fluctuate between \( p(0)e^{-c} \) and \( p(0)e^c \). Furthermore, the equilibrium value of population \( \bar{p} \) is obtained when the rate of change \( \frac{dp}{dt} \) is zero. This would be, if
\[
\frac{dp}{dt} = cp(t) \cos t = 0 \quad \text{or} \quad \cos t = 0
\]

This can occur at \( t = \frac{\pi}{2} \) or \( t = \frac{3\pi}{2} \) when \( \sin \frac{\pi}{2} = 1 \) and \( \sin \frac{3\pi}{2} = -1 \). Therefore, the population function takes values \( p(t) = p(0)e^c \) and \( p(t) = p(0)e^{-c} \) in a stationary state. Thus, it is oscillating between these equilibrium values.

7. If the demand and supply functions for a commodity are given by \( q_d = \alpha - \beta p \) and \( q_s = \gamma \sin nt \), use the model of market price dynamics to determine \( p(t) \) and analyze its behavior as \( t \) increases.

**Solution:**

In accordance with the model,
\[
\frac{dp}{dt} = j(q_d - q_s)
\]

Substituting for the demand and supply functions,
\[
\frac{dp}{dt} = j(\alpha - \beta p - \gamma \sin nt)
\]

\[
\frac{dp}{dt} + \beta jp = j(\alpha - \gamma \sin nt)
\]

\[
p(t) = \left[ p(0) - \frac{\alpha - \gamma \sin nt}{\beta} \right] e^{-\beta t} + \frac{\alpha - \gamma \sin nt}{\beta}
\]

As we would normally expect \( \beta \) and the adjustment coefficient \( j \) to be positive, the time path of market price is convergent with time. The equilibrium value of price is

\[
\bar{p} = \frac{\alpha - \gamma \sin nt}{\beta}
\]
Since the equilibrium price contains a circular function, the values of which fluctuate between -1 and 1, the price is oscillating between two stationary values, $\frac{\alpha - \gamma}{\beta}$ and $\frac{\alpha + \gamma}{\beta}$.

8. Consider the following demand and supply functions for a commodity:

\[ q_d = 22 - 3p - 4p' + p^* \]
\[ q_s = -3 + 2p \]

with initial conditions $p(0) = 7$ and $p'(0) = 4$. Find the time path of market price $p(t)$ and determine whether price converges to its intertemporal equilibrium.

Solution:

The concrete values of the parameters are
\[ \alpha = 22 \quad \beta = 3 \quad \gamma = 3 \quad \delta = 2 \quad u = -4 \quad v = 1 \]

Equating the two market forces,
\[ 22 - 3p - 4p' + p^* = -3 + 2p \]
\[ p^* - 4p' - 5p = -25 \]

We can find the intertemporal equilibrium given by the particular integral either directly from the differential equation or by the formula

\[ p_p = \frac{\alpha + \gamma}{\beta + \delta} = \frac{22 + 3}{3 + 2} = 5 \]

Since $v > 0$, the only feasible case is that of distinct real roots. Solving for the characteristic roots by the formula

\[ r_{1,2} = \frac{1}{2} \left[ \frac{-u \pm \sqrt{\left(\frac{u}{v}\right)^2 + 4\left(\frac{\beta + \delta}{v}\right)}}{v} \right] = \frac{1}{2} \left[ 4 \pm \sqrt{16 + 4(3 + 2)} \right] = \frac{1}{2} (4 \pm 4) = 5, -1 \]

Thus, the general solution is

\[ p(t) = p_c + p_p = A_1 e^{5t} + A_2 e^{-t} + 5 \]

To definitize the constants $A_1$ and $A_2$, we use the initial conditions:

\[ p(0) = A_1 e^{0} + A_2 e^{0} + 5 = 7 \quad \text{or} \quad A_1 + A_2 = 2 \]

Differentiating to find the first derivative of price,

\[ p'(t) = 5A_1 e^{5t} - A_2 e^{-t} \quad \text{or} \quad p'(0) = 5A_1 - A_2 = 4 \]

which gives $A_1 = A_2 = 1$, and the definite solution is

\[ p(t) = e^{5t} + e^{-t} + 5 \]

Since one of the characteristic roots is positive ($r_1 = 5$), the intertemporal equilibrium of 5 is dynamically unstable.

9. The following market model is given:
\[ q_d = 38 - 3p - 6p' + 2p'' \]
\[ q_s = -10 + 5p \]

where \( p(0) = 11 \) and \( p'(0) = 5 \). Find the time path of market price \( p(t) \) assuming that the market clears at every point of time.

**Solution:**

We can solve again by substituting the values for the different parameters, but one other way is to solve the characteristic equation.

\[ 38 - 3p - 6p' + 2p'' = -10 + 5p \]

\[ 2p'' - 6p' - 8p = -48 \]

\[ p'' - 3p' - 4p = -24 \]

\[ a_1 = -3 \quad a_2 = -4 \quad b = -24 \]

The intertemporal equilibrium is given by the particular integral:

\[ p_p = \frac{b}{a_2} = \frac{-24}{-4} = 6 \]

Here \( u < 0 \) and \( v > 0 \), so only the case of distinct real roots is possible. Solving for the characteristic roots by the formula

\[ r^2 + a_1r + a_2 = 0 \quad \text{or} \]

\[ r^2 - 3r - 4 = 0 \]

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{1}{2} (3 \pm \sqrt{9 + 16}) = \frac{1}{2} (3 \pm 5) = 4, -1 \]

which produces one positive root so the intertemporal equilibrium of 6 for price is dynamically unstable. The general solution becomes

\[ p(t) = A_1 e^{4t} + A_2 e^{-t} + 6 \]

To find the constants \( A_1 \) and \( A_2 \), we set \( t = 0 \).

\[ p(0) = A_1 e^0 + A_2 e^0 + 6 = 11 \quad \text{or} \quad A_1 + A_2 = 5 \]

\[ p'(t) = 4A_1 e^{4t} - A_2 e^{-t} \quad \text{or} \quad 4A_1 - A_2 = 5 \]

so \( A_1 = 2 \) and \( A_2 = 3 \). The definite solution is

\[ p(t) = 2e^{4t} + 3e^{-t} + 6 \]

**10.** For the following demand and supply functions

\[ q_d = 8 - p - 3p' - p'' \]
\[ q_s = -2 + p \]

where \( p(0) = 8 \) and \( p'(0) = -4 \), express the general and definite solution for price \( p(t) \) assuming market equilibrium at every moment.
Solution:

\[ 8 - p - 3p' - p'' = -2 + p \]

\[ p'' + 3p' + 2p = 10 \]

\[ a_1 = 3 \quad a_2 = 2 \quad b = 10 \quad u = -3 \quad v = -1 \]

\[ p_p = \frac{b}{a_2} = \frac{10}{2} = 5 \]

Here \( u, v < 0 \), so the intertemporal equilibrium of 5 must be dynamically stable and the time path of price should be convergent. The characteristic equation is

\[ r^2 + 3r + 2 = 0 \]

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{1}{2} (-3 \pm \sqrt{9 - 8}) = \frac{1}{2} (-3 \pm 1) = -2, -1 \]

As we expected, both characteristic roots are negative, which ensures the dynamic stability of the time path of price. Solving further,

\[ p(t) = A_1 e^{-t} + A_2 e^{-2t} + 5 \]

To find the constants \( A_1 \) and \( A_2 \), we set \( t = 0 \).

\[ p(0) = A_1 e^0 + A_2 e^0 + 5 = 8 \quad \text{or} \quad A_1 + A_2 = 3 \]

\[ p'(t) = -A_1 e^{-t} - 2A_2 e^{-t} \quad \text{or} \quad -A_1 - 2A_2 = -4 \]

which results in \( A_1 = 2 \) and \( A_2 = 1 \). Thus, the definite solution is

\[ p(t) = 2e^{-t} + e^{-2t} + 5 \]

11. If the time path of price stems from the equation \( p'' + 4p' + 4p = 12 \), what is the general solution for \( p(t) \)? Is the time path of price likely to be convergent or divergent? Assume that \( p(0) = 4 \) and \( p'(0) = 1 \) to find the definite solution.

Solution:

We can immediately determine the parameters:

\[ a_1 = 4 \quad a_2 = 4 \quad b = 12 \]

\[ p_p = \frac{b}{a_2} = \frac{12}{4} = 3 \]

We must have \( u = -4 \) and \( v = -1 \) in order for the right signs of the other parameters to obtain (check why with some hypothetical demand and supply functions). Then, the intertemporal equilibrium of 3 must be dynamically stable. The characteristic equation is

\[ r^2 + 4r + 4 = 0 \]

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = (-2 \pm \sqrt{4 - 4}) = -2 \]
We get case 2 of a single real root. Since this root of -2 is negative, this ensures the dynamic stability for the price function. Therefore, the general solution is

\[ p(t) = A_1 e^{-2t} + A_2 t e^{-2t} + 3 \]

To find the constants \( A_1 \) and \( A_2 \), we set \( t = 0 \).

\[
p(0) = A_1 e^0 + 3 = 4 \quad \text{or} \quad A_1 = 1
\]

\[
p'(t) = -2A_1 e^{-2t} + A_2 e^{-2t} - 2A_2 t e^{-t} \quad \text{or} \quad -2A_1 + A_2 = 1
\]

so \( A_1 = 1 \) and \( A_2 = 3 \). Therefore, the definite solution is

\[ p(t) = e^{-2t} + 3t e^{-2t} + 3 \]

12. Consider the following demand and supply functions:

\[ q_d = 20 - 5p - 6p' - p^* \]
\[ q_s = -7 + 4p \]

Assume market equilibrium at any point in time and find the time path of \( p(t) \).

Solution:

Equating demand and supply,

\[
p^* + 6p' + 9p = 27
\]

\[ p_p = \frac{b}{a} = 3 \]

Here \( u = -6 \) and \( v = -1 \); therefore, the intertemporal equilibrium of 3 must be dynamically stable.

The characteristic equation is \( r^2 + 6r + 9 = 0 \) which can be written as \((r + 3)^2 = 0\) or \( r = -3 \). Since it is a negative single root we get, we can be sure that the time path of the price function is convergent.

The general solution is

\[ p(t) = A_1 e^{-3t} + A_2 e^{-3t} + 3 \]

13. In the following market model,

\[ q_d = 20 - 3p - 4p' - p^* \]
\[ q_s = -5 + 2p \]

where \( p(0) = 7 \) and \( p'(0) = 1 \), express the general and definite solution for price \( p(t) \) assuming market equilibrium at every moment.

Solution:

\[ 20 - 3p - 4p' - p^* = -5 + 2p \]

\[ p^* + 4p' + 5p = 25 \]

\[ p_p = \frac{b}{a} = \frac{25}{5} = 5 \]

Here \( u = -4 \) and \( v = -1 \), so all the three cases are feasible. The intertemporal equilibrium is dynamically stable. The characteristic equation is
\[ r^2 + 4r + 5 = 0 \]
\[ r_{1,2} = -2 \pm \sqrt{4 - 5} = -2 \pm \sqrt{-1} = -2 \pm i \]

Here the roots are a pair of conjugate complex numbers of the type \( r_{i,2} = m \pm ni \) where the real part is \( m = -2 \) and the imaginary is \( n = 1 \). As we expected, a negative real part ensures the dynamic stability of the time path of price. Using the formula for complex numbers, we can write the general solution

\[ p(t) = e^{-2t}(B_1 \cos t + B_2 \sin t) + 5 \]

To definitize the constants, we set \( t = 0 \).

\[ p(0) = e^0(B_1 \cos 0 + B_2 \sin 0) + 5 = 7 \]
\[ \cos 0 = 1 \quad \sin 0 = 0 \quad \text{or} \quad B_1 = 2 \]

\[ p'(0) = -2e^{-2t}(B_1 \cos t + B_2 \sin t) + e^{-2t}(-B_1 \sin t + B_2 \cos t) \]

\[ p'(0) = -2e^0(B_1 \cos 0 + B_2 \sin 0) + e^0(-B_1 \sin 0 + B_2 \cos 0) = -2B_1 + B_2 = 1 \]

which gives \( B_1 = 2 \) and \( B_2 = 5 \)

\[ p(t) = e^{-2t}(2 \cos t + 5 \sin t) + 5 \]

With circular functions in the solution, the time path is periodic fluctuation with a period \( \frac{2\pi}{n} \) or \( 2\pi \). The price performs a full cycle every time \( t \) increases by \( 2\pi \). Since \( m = -2 \), the fluctuation is damped, and price converges to the intertemporal equilibrium of 5 in a cyclical and damped way.

14. Given the demand and supply functions on a market for a commodity

\[ q_d = 11 - p + p' + 3p^* \]
\[ q_s = -4 + 4p - p' + 5p^* \]

where \( p(0) = 7 \) and \( p'(0) = 5 \), determine whether price fluctuates in time. Is the fluctuation explosive or damped? Assume the market clears at every moment.

Solution:

\[ 11 - p + p' + 3p^* = -4 + 4p - p' + 5p^* \]
\[ 2p^* - 2p' + 5p = 15 \]
\[ p^* - p' + 2.5p = 7.5 \]
\[ p_p = \frac{7.5}{2.5} = 3 \]

The characteristic equation is

\[ r^2 - r + 2.5 = 0 \]

\[ r_{1,2} = \frac{1}{2}(1 \pm \sqrt{1 - 10}) = \frac{1}{2}(1 \pm 3i) = \frac{1}{2} \pm \frac{3}{2}i \]

The roots are a pair of complex numbers with a real part \( m = \frac{1}{2} \) and an imaginary one \( n = \frac{3}{2} \). Hence, the general solution is
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\[ p(t) = e^{\frac{t}{2}} \left( B_1 \cos \frac{3t}{2} + B_2 \sin \frac{3t}{2} \right) + 3 \]

To definitize the constants, we set \( t = 0 \).

\[ p(0) = e^0 (B_1 \cos 0 + B_2 \sin 0) + 3 = 7 \quad \cos 0 = 1 \quad \sin 0 = 0 \quad \text{or} \quad B_1 = 4 \]

\[ p'(t) = \frac{e^{t/2}}{2} \left( B_1 \cos \frac{3t}{2} + B_2 \sin \frac{3t}{2} \right) + \frac{e^{t/2}}{2} \left( -\frac{3B_1}{2} \sin \frac{3t}{2} + \frac{3B_2}{2} \cos \frac{3t}{2} \right) \]

\[ p'(0) = \frac{e^0}{2} (B_1 \cos 0 + B_2 \sin 0) + e^0 \left( -\frac{3B_1}{2} \sin 0 + \frac{3B_2}{2} \cos 0 \right) = \frac{B_1}{2} + \frac{3B_2}{2} = 5 \]

\[ B_1 + 3B_2 = 10 \quad \text{which results in} \quad B_1 = 4 \quad \text{and} \quad B_2 = 2 \]

\[ p(t) = e^{\frac{t}{2}} \left( 4 \cos \frac{3t}{2} + 2 \sin \frac{3t}{2} \right) + 3 \]

The price function fluctuates periodically with a period \( \frac{2\pi}{n} = \frac{2(3)\pi}{2} = 3\pi \). The price performs a full cycle every time \( t \) increases by \( 3\pi \). Since the real part is positive or \( m = \frac{1}{2} \), the fluctuation is explosive, and price diverges from its intertemporal equilibrium of 3 cyclically.

15. Consider an expanded market equilibrium model that takes into account both buyers’ and sellers’ expectations of price change such that

\[ q_u = \alpha - \beta p + u_1 p' + v_1 p'' \quad \alpha, \beta > 0 \]

\[ q_s = -\gamma + \delta p + u_2 p' + v_2 p'' \quad \gamma, \delta > 0 \]

Assuming the market is always in equilibrium, express the time path of price. Find also its intertemporal equilibrium and determine how it is influenced by the expectations of market participants. Under what circumstances could a single real root and periodic fluctuation be ruled out?

**Solution:**

Equating demand and supply,

\[ \alpha - \beta p + u_1 p' + v_1 p'' = -\gamma + \delta p + u_2 p' + v_2 p'' \]

\[ (v_1 - v_2) p'' + (u_1 - u_2) p' - (\beta + \delta) p = -\alpha - \gamma \]

Let \( v = v_1 - v_2 \) and \( u = u_1 - u_2 \). Normalizing the equation, we get

\[ \frac{p''}{v} + \frac{u}{v} p' - \frac{(\beta + \delta)}{v} p = -\frac{\alpha + \gamma}{v} \]

As with the simple market equilibrium model, the intertemporal equilibrium is

\[ p_p = \frac{\alpha + \gamma}{\beta + \delta} \]

We can see that it does not contain any of the coefficients \( u_i, v_i \ (i = 1, 2) \) that reflect the expectations of the market participants. Hence, the intertemporal equilibrium price does not depend on the expectations of either market group, but depends solely on current price. This is quite logical; the time path shows a changing equilibrium price at any moment in time. This dynamically changing equilibrium depends on people’s expectations. The intertemporal equilibrium price, though, does not
depend on trends or short-term expectations and is the normal-level price throughout a long period of
time. Since \( a_1 = \frac{u}{v} \) and \( a_2 = -\frac{\beta + \delta}{v} \), the characteristic roots are
\[
r_{1,2} = \frac{1}{2} \left[ \frac{-u}{v} \pm \sqrt{\left(\frac{u}{v}\right)^2 + 4\left(\frac{\beta + \delta}{v}\right)} \right]
\]
The roots are identical with the simple model but note that here \( u \) and \( v \) are products of both the
demand and the supply functions. Thus, the general solution in the three known cases is

**Case 1.** Distinct real roots
\[
\left(\frac{u}{v}\right)^2 > -4\left(\frac{\beta + \delta}{v}\right)
\]
\[
p(t) = p_e + p_r = A_1e^{rt} + A_2e^{ct} + \frac{\alpha + \gamma}{\beta + \delta}
\]

**Case 2.** Single real root
\[
\left(\frac{u}{v}\right)^2 = -4\left(\frac{\beta + \delta}{v}\right)
\]
The single root is \( r = -\frac{a_1}{2} = -\frac{u}{2v} \).
\[
p(t) = A_1e^{\frac{mt}{2v}} + A_2te^{\frac{mt}{2v}} + \frac{\alpha + \gamma}{\beta + \delta}
\]

**Case 3.** Complex roots
\[
\left(\frac{u}{v}\right)^2 < -4\left(\frac{\beta + \delta}{v}\right)
\]
The characteristic roots are a pair of conjugate complex numbers and
\[
m = -\frac{u}{2v} \quad \text{and} \quad n = \frac{1}{2} \sqrt{-4\left(\frac{\beta + \delta}{v}\right) - \left(\frac{u}{v}\right)^2}
\]
\[
p(t) = e^{\frac{mt}{2v}}(B_1\cos nt + B_2\sin nt) + \frac{\alpha + \gamma}{\beta + \delta}
\]
To rule out the last two cases, we should have \( v > 0 \); that is, \( v_1 - v_2 > 0 \) or \( v_1 > v_2 \). Then \( -4\left(\frac{\beta + \delta}{v}\right) \)
is always negative, so cases 2 and 3 are impossible.

**16.** With the price of real estate rising at an increasing rate in Bulgaria \( (p', p'' > 0) \) at the time of the
real estate boom, the construction business felt stimulated to supply many more new buildings.
Expecting prices to rise further and profits to grow higher, builders built more and more intensively.
The model, with consumer expectations ignored, becomes
\[
q_d = \alpha - \beta p \quad \quad \alpha, \beta > 0
\]
\[
q_s = -\gamma + \delta p + up' + vp' \quad \quad \gamma, \delta > 0 \quad u, v > 0
\]
Assuming the market is always in equilibrium, express the time path of price. Find also its
intertemporal equilibrium. Which of the three familiar cases are possible and dynamically stable?
Solution:

Assuming equilibrium on the housing market,

\[ \alpha - \beta p = -\gamma + \delta p + up' + vp'' \]

\[ vp'' + up' + (\beta + \delta) p = \alpha + \gamma \]

\[ p'' + \frac{u}{v} p' + \frac{(\beta + \delta)}{v} p = \frac{\alpha + \gamma}{v} \]

As with the simple market equilibrium model, the intertemporal equilibrium is

\[ p_p = \frac{\alpha + \gamma}{\beta + \delta} \]

which again is independent of the expectations of the Bulgarian builders. Since \( a_1 = \frac{u}{v} \) and \( a_2 = \frac{\beta + \delta}{v} \), the characteristic roots are

\[ r_{1,2} = \frac{1}{2} \left[ -\frac{u}{v} \pm \sqrt{\left(\frac{u}{v}\right)^2 - 4\left(\frac{\beta + \delta}{v}\right)} \right] \]

Case 1. Distinct real roots

\( \left(\frac{u}{v}\right)^2 > 4\left(\frac{\beta + \delta}{v}\right) \)

\[ p(t) = p_c + p_p = A_1e^{\alpha t} + A_2e^{\alpha t} + \frac{\alpha + \gamma}{\beta + \delta} \]

Case 2. Single real root

\( \left(\frac{u}{v}\right)^2 = 4\left(\frac{\beta + \delta}{v}\right) \)

The root is \( r = -\frac{a_1}{2} = -\frac{u}{2v} \).

\[ p(t) = A_1e^{\frac{u}{2v}t} + A_2e^{\frac{u}{2v}t} + \frac{\alpha + \gamma}{\beta + \delta} \]

Case 3. Complex roots

\( \left(\frac{u}{v}\right)^2 < 4\left(\frac{\beta + \delta}{v}\right) \)

The solution is

\[ p(t) = e^{\frac{u}{2v}t} (B_1 \cos nt + B_2 \sin nt) + \frac{\alpha + \gamma}{\beta + \delta} \]

In view of the positive \( u \) and \( v \), all the three cases are possible. Furthermore, in case 1 both characteristic roots are negative. (Can you tell why?) So is the single root in the second case. Lastly, in case 3 the real part of the characteristic roots \( h \) is negative. This is sufficient for the time path of price to be dynamically stable in all the three cases. Hence, the condition \( u, v > 0 \) denoting the optimistic expectations of the Bulgarian builders guarantees the dynamic stability of price of real estate in the Bulgarian construction market.
17. The national income of a country is changing according to the equation \( Y''(t) + 8Y'(t) = 32 \). Find the time path of national income and its intertemporal equilibrium, if it exists. Say whether income is converging to this equilibrium. Assume initial conditions of \( Y(0) = 5 \) and \( Y'(0) = -12 \). Express the national income in the third period.

**Solution:**

This is a second-order differential equation for which

\[
a_1 = 8 \quad a_2 = 0 \quad \text{and} \quad b = 32
\]

Using the formula for the particular integral, we have

\[
Y_p = \frac{b}{a_1} t = \frac{32}{8} t = 4t
\]

Since \( Y_p \) is a function of \( t \), this is not a stationary, but a moving equilibrium. Furthermore, since every next moment of time \( t \) is bigger, the equilibrium value is expected to grow. The characteristic equation is

\[
r^2 + a_1 r = 0
\]

\[
r(r + a_1) = 0
\]

which gives the characteristic roots \( r_1 = 0 \), \( r_2 = -a_1 = -8 \). Thus, the general solution for national income is

\[
Y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + Y_p = A_1 e^{0t} + A_2 e^{-8t} + 4t = A_1 + A_2 e^{-8t} + 4t
\]

To specify the constants,

\[
Y(0) = A_1 + A_2 e^{-8(0)} + 4(0) = A_1 + A_2 = 5
\]

\[
Y'(t) = -8A_2 e^{-8t} + 4
\]

\[
Y'(0) = -8A_2 e^{0} + 4 = -8A_2 + 4 = -12 \quad \text{or} \quad A_2 = 2 \quad \text{and} \quad A_1 = 3
\]

The definite solution is

\[
Y(t) = 3 + 2e^{-8t} + 4t
\]

Since the nonzero characteristic root is negative, national income converges to its intertemporal equilibrium – but it is a moving equilibrium. Furthermore, the income of the nation is increasing with time starting from a positive initial level of 5. In the third period, the income in this growing economy is

\[
Y(3) = 3 + 2e^{-8(3)} + 4(3) = 15 + 2e^{-24}
\]

18. The increase in the rate of change of national income of a country is given by \( Y''(t) = 14 \). Find the time path of national income and its intertemporal equilibrium, if it exists. Assume that in the initial moment \( t = 0 \) the national income is 15. How is the income of this country changing with time? Check the first and the second derivative of the national income function.

**Solution:**

This is a second-order differential equation for which

\[
a_1 = 0 \quad a_2 = 0 \quad \text{and} \quad b = 14
\]
Therefore, using the formula for the particular integral, we have

\[
Y_p = \frac{b}{2} t^2 = \frac{14}{2} t^2 = 7t^2
\]

Since \( Y_p \) is a function of \( t \), this is not a stationary but a moving equilibrium, and it is rapidly increasing with time. Since \( r_1 = r_2 = 0 \), the general solution becomes

\[
Y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + Y_p = A_1 e^{0 t} + A_2 e^{0 t} + 7t^2 = A_1 + A_2 + 7t^2
\]

In the initial moment, \( Y(0) = 15 \). Hence,

\[
Y(0) = A_1 + A_2 + 7(0) = A_1 + A_2 = 15
\]

We can write the national income function as

\[
Y(t) = 15 + 7t^2
\]

The national income takes a higher value in every next period. Thus, in the first period it is \( Y(1) = 22 \), while in the second it is \( Y(2) = 43 \). From the definite solution, we can easily check the two consecutive derivatives \( Y'(t) = 14t \) and \( Y''(t) = 14 \), which prove our calculations correct.

19. Let \( u(w) \) be the utility function over wealth \( w \). At any wealth level \( w \), the Arrow-Pratt measure of absolute risk aversion is the percent rate of change of marginal utility \( u' \) at \( w \) so it equals \( \frac{u''(w)}{u'(w)} \).

Assume a utility function that has a constant absolute risk aversion \( a \). Find the general form of that utility function. Check that, indeed, it has a constant absolute risk aversion. What is the condition for marginal utility \( u' \) to be positive?

**Solution:**

To find the particular utility function we have to solve the second-order differential equation

\[
-a \frac{u'(w)}{u^*(w)} = a \quad \text{or} \quad u^*(w) + au'(w) = 0
\]

Here we have \( a_1 = a \), \( a_2 = 0 \), and \( b = 0 \). Hence, the particular integral is \( u_p = 0 \) and the characteristic equation is \( r^2 + ar = 0 \) which gives roots \( r_{1,2} = 0, -a \). Thus, the general solution is

\[
u(w) = A_1 + A_2 e^{-aw}
\]

To check for absolute risk aversion, we differentiate sequentially:

\[
u'(w) = -aA_2e^{-aw} \quad \text{and} \quad u''(w) = a^2 A_2 e^{-aw}
\]

Taking the ratio of the two derivatives with a minus, we get a constant absolute risk aversion of exactly \( a \):

\[
\frac{-u^*(w)}{u'(w)} = \frac{a^2 A_2 e^{-aw}}{aA_2 e^{-aw}} = a
\]
In order for marginal utility to be positive for a positive total utility \( u'(w) = -aA_w e^{-aw} > 0 \) implies that the constant \( A_w \) be negative.

20. In relation to the previous problem, assume now that the absolute risk aversion function is non-constant taking the specific form \( a(w) = bw \) where \( b \) is a positive parameter. Thus, when the individual’s wealth is increasing his risk aversion increases as well at the constant rate \( b \). Find the marginal utility of wealth \( u'(w) \) and set the condition for it to be positive.

**Solution:**

\[
\frac{u''(w)}{u'(w)} = a(w) = bw
\]

\[u''(w) + bwu'(w) = 0\]

which is a second-order differential equation in marginal utility of wealth \( u'(w) \). Setting \( u_w = u'(w) \),

\[u''_w + bwu_w = 0\]

and resorting to separation of variables

\[\frac{u'_w}{u_w} = -bw\]

\[
\int \frac{u'_w}{u_w} dw = -\int bw dw
\]

\[\ln u_w = -\frac{bw^2}{2} + c\]

and taking the antilog of both sides

\[u_w = e^{-\frac{bw^2}{2}} e^c = Ce^{-\frac{bw^2}{2}}\]

\(u_w > 0\) implies \(C > 0\)

Integrating marginal utility \( u_w \) would give the total utility function \( u(w) = \int Ce^{-\frac{bw^2}{2}} dw \).

21. The consumption function of an individual grows with time according to the equation

\[C''(t) + 9C'(t) + 14C = 42\]. Find the time path of this person’s consumption given that \( C(0) = 6 \) and \( C'(0) = -16 \). What is the amount of his consumption in the second period?

**Solution:**

For the particular integral, we have

\[C_p = \frac{42}{14} = 3\]

The characteristic roots are

\[r_{1,2} = \frac{-9 \pm \sqrt{81 - 4(14)}}{2} = \frac{-9 \pm 5}{2} = -2, -7\]

Since both characteristic roots are negative, this implies dynamic stability for the time path of the consumer. For the general solution

\[C(t) = A_1 e^{2t} + A_2 e^{-7t} + C_p = A_1 e^{-2t} + A_2 e^{-7t} + 3\]
To specify the constants,
\[ C(0) = A_1 e^0 + A_2 e^0 + 3 = A_1 + A_2 + 3 = 6 \quad A_1 + A_2 = 3 \]
\[ C'(t) = -2A_1 e^{-2t} - 7A_2 e^{-7t} \]
\[ C''(0) = -2A_1 e^0 - 7A_2 e^0 = -2A_1 - 7A_2 = -16 \]
which gives \( A_1 = 1 \) and \( A_2 = 2 \). The definite solution is
\[ C(t) = e^{-2t} + 2e^{-7t} + 3 \]
The consumption of the individual converges to the intertemporal equilibrium of 3. In the second period the consumption is
\[ C(2) = e^{-2(2)} + 2e^{-7(2)} + 3 = e^{-4} + 2e^{-14} + 3 \]

22. The following differential equation gives the aggregate savings of a country
\[ S''(t) + 6S'(t) + 5S = 10 \] Establish the time path of aggregate savings for \( S(0) = -4 \) and \( S'(0) = 2 \). Is it dynamically stable? How do savings change from their initial level?

**Solution:**

For the intertemporal equilibrium of savings, we have
\[ S_p = \frac{10}{5} = 2 \]

The characteristic roots are
\[ r_{1,2} = -3 \pm \sqrt{9 - 5} = -3 \pm 2 = -5, -1 \]
Since both characteristic roots are negative, the time path of savings is dynamically stable and converges to the equilibrium of 2. The general solution is
\[ S(t) = A_1 e^{-5t} + A_2 e^{-t} + 2 \]
To specify the constants,
\[ S(t) = A_1 e^0 + A_2 e^0 + 2 = A_1 + A_2 + 2 = -4 \quad A_1 + A_2 = -6 \]
For the first derivative, we have
\[ S'(t) = -5A_1 e^{-5t} - A_2 e^{-t} \]
\[ S'(0) = -5A_1 e^0 - A_2 e^0 = -5A_1 - A_2 = 2 \]
which gives \( A_1 = 1 \) and \( A_2 = -7 \). Therefore, the savings function is
\[ S(t) = e^{-5t} - 7e^{-t} + 2 \]
We can check the parental function in the initial moment
\[ S(0) = e^0 - 7e^0 + 2 = 1 - 7 + 2 = -4 \]
which is the initial value of savings. Since this initial value is negative, we can conclude that the nation borrows in the beginning. Since the time path is convergent, as time passes the savings become positive and converge to the equilibrium level.
23. The aggregate savings of a nation change according to the equation \( S''(t) + 6S'(t) + 9S = 36 \). Find the time path of the aggregate savings function if \( S(0) = -2 \) and \( S'(0) = 16 \). What is the intertemporal equilibrium for savings, and is the nation moving to it or diverging from it? Interpret the results economically. Differentiate the definite solution to check the validity of your calculations.

Solution:
For the intertemporal equilibrium of savings, we have
\[
S_p = \frac{36}{9} = 4
\]
The characteristic roots are \( r_{1,2} = -3 \pm \sqrt{9 - 9} = -3 \). We obtain a single root that is negative, implying dynamic stability for the time path of savings. Thus, the nation is moving in the direction of this intertemporal equilibrium for savings. The general solution can be expressed as
\[
S(t) = A_1 e^{rt} + A_2 te^{rt} + S_p = A_1 e^{-3t} + A_2 te^{-3t} + 4
\]
To specify the constants,
\[
S(0) = A_1 e^0 + A_2 (0)e^0 + 4 = -2 \quad \quad A_1 = -6
\]
Furthermore, for the first derivative from the general solution, we have
\[
S'(t) = -3A_1 e^{-3t} + A_2 e^{-3t} - 3A_2 e^{-3t}
\]
\[
S'(0) = -3A_1 e^0 + A_2 e^0 + 0 = -3A_1 + A_2 = 16
\]
which gives \( A_1 = -6 \) and \( A_2 = -2 \). The definite solution becomes
\[
S(t) = -6e^{-3t} - 2te^{-3t} + 4
\]
We can check the parental function and its first derivative using this definite solution.
\[
S(0) = -6e^0 - 2(0)e^0 + 4 = -6 + 4 = -2
\]
which is the initial level of savings. Note that since aggregate savings are initially negative, the nation must be a net borrower in the beginning. With the passage of time, savings become positive and converge to the equilibrium level of 4. For the first derivative from the definite solution, we have
\[
S'(t) = 18e^{-3t} + 36e^{-3t} = 16e^{-3t} + 6te^{-3t}
\]
Setting \( t = 0 \),
\[
S'(0) = 16e^0 + 6(0)e^0 = 16
\]
Checking for \( S(0) \) and \( S'(0) \), we get exactly the values given, which proves our computations correct.

24. Let the demand and supply functions be the well-known
\[
q_d = \alpha - \beta p + up' + vp^* \quad \alpha, \beta > 0
\]
\[
q_s = -\gamma + \delta p \quad \gamma, \delta > 0
\]
It is given that the market does not always clear but adjusts according to the relationship
\[
\frac{dp}{dt} = j(q_d - q_s) \quad \text{for} \quad j > 0 \quad \text{where the change in the price depends on the level of excess demand by the amount of a positive adjustment coefficient} \ j. \ \text{Find and compare the intertemporal equilibrium price and the market-clearing equilibrium price. State the conditions for dynamic stability.}
Solution:
Writing the differential equation,
\[ p' = j(\alpha - \beta p + up' + vp^* + \gamma - \delta p) \]
\[ p' = j(\alpha + \gamma) - j(\beta + \delta)p + jup' + jvp^* \]
\[ jvp^* + (ju - 1)p' - j(\beta + \delta)p = -j(\alpha + \gamma) \]

Normalizing,
\[ p^* + \frac{(ju - 1)}{jv}p' - \frac{(\beta + \delta)}{v}p = -\frac{(\alpha + \gamma)}{v} \]
\[ p^* + \frac{(u - 1/j)}{v}p' - \frac{(\beta + \delta)}{v}p = -\frac{(\alpha + \gamma)}{v} \]

We can easily find the intertemporal equilibrium by the well-known formula \( p_p = \frac{\alpha + \gamma}{\beta + \delta} \). The intertemporal equilibrium is identical to that under market clearance in every instant. Whether the market is continually in equilibrium or not, the intertemporal equilibrium stays the same. Furthermore, we see that the intertemporal equilibrium, unlike the market-clearing one, does not depend on expectation coefficients \( u \) and \( v \). Here we have \( a_1 = \frac{(u - 1/j)}{v} \) and \( a_2 = \frac{(\beta + \delta)}{v} \). There are three possibilities, as with the simple market equilibrium model:

**Case 1.** Distinct real roots
\[ \left( \frac{u - 1/j}{v} \right)^2 > -4 \left( \frac{\beta + \delta}{v} \right) \]
The characteristic roots are
\[ r_{1,2} = \frac{1}{2} \left[ \frac{u - 1/j}{v} \pm \sqrt{ \left( \frac{u - 1/j}{v} \right)^2 + 4 \left( \frac{\beta + \delta}{v} \right) } \right] \]
The general solution is
\[ p(t) = p_c + p_p = A_1e^{rt} + A_2e^{r't} + \frac{\alpha + \gamma}{\beta + \delta} \]

**Case 2.** Single real root
\[ \left( \frac{u - 1/j}{v} \right)^2 = -4 \left( \frac{\beta + \delta}{v} \right) \]
\[ r = -\frac{a_1}{2} = -\frac{(u - 1/j)}{2v} \]
The general solution is
\[ p(t) = A_1e^{\frac{(u - 1/j)t}{2v}} + A_2te^{\frac{(u - 1/j)t}{2v}} + \frac{\alpha + \gamma}{\beta + \delta} \]

**Case 3.** Complex roots
\[ \left( \frac{u - 1/j}{v} \right)^2 < -4 \left( \frac{\beta + \delta}{v} \right) \]
The characteristic roots are the complex numbers for which
\[ m = \frac{a_1}{2} = -\frac{(u-1/j)}{2v} \quad \text{and} \quad n = \frac{\sqrt{4a_2-a_1^2}}{2} = \frac{1}{2} \sqrt{-4\left(\frac{\beta + \delta}{v}\right) - \left(\frac{u-1/j}{v}\right)^2} \]

For the general solution,
\[ p(t) = e^{-\frac{(u-1/j)t}{2v}} (B_1 \cos nt + B_2 \sin nt) + \frac{\alpha + \gamma}{\beta + \delta} \]

The condition for dynamic stability is \( v < 0 \) and \( u < \frac{1}{j} \). All three cases considered are possible.

Under the above conditions, all roots, distinct or repeated, are negative. In the case of complex numbers, \( m \) is negative too. Therefore, the dynamic stability of the price function is ensured when \( v < 0 \) and \( v < \frac{1}{j} \).

25. The following three equations of the Phillips relation model are given:
\[ \dot{p} = 3 - 2U + \pi \\
\frac{d\pi}{dt} = \frac{1}{2} (\dot{p} - \pi) \\
\frac{dU}{dt} = -(m - \dot{p}) \]

Find the time path of the expected rate of inflation and determine whether it converges to its intertemporal equilibrium. Express also the time paths of the price level and the unemployment rate. Discuss the relationship between inflation and unemployment in the short and in the long run.

**Solution:**

We have the following values of the parameters:
\[ \beta = 2 \quad j = \frac{1}{2} \quad h = 1 \quad \text{and} \quad k = 1 \]

Using the formulas for the coefficients \( a_1, a_2 \) and \( b \), we obtain, respectively,
\[ a_1 = \beta k + j(1-h) = 2(1) + \frac{1}{2}(0) = 2 \]
\[ a_2 = j\beta k = \frac{1}{2} (2)(1) = 1 \quad b = j\beta km = \dot{m} \]

The intertemporal equilibrium of the expected rate of inflation is \( \pi_p = \dot{m} \). For the characteristic roots,
\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{1}{2} \left(-2 \pm \sqrt{4 - 4}\right) = -\frac{2}{2} = -1 \]

or we have the single-root case, so the time path of expected inflation can be written as
\[ \pi(t) = A_1 e^{-t} + A_2 te^{-t} + \dot{m} \]

Since the root is negative, the time path of expected inflation is convergent to the growth rate of nominal money. Knowing the time path of \( \pi \), we can find that of \( \dot{p} \). From the relationship
\[ \dot{p} = \frac{1}{j} \frac{d\pi}{dt} + \pi \quad \text{we get} \]

\[ \dot{p} = 2 \frac{d\pi}{dt} + \pi \]

Differentiating \( \pi \),
\[ \pi'(t) = -A_1 e^{-t} - A_2 t e^{-t} + A_2 e^{-t} \]

and substituting in the expression,
\[ \dot{p} = 2(-A_1 e^{-t} - A_2 t e^{-t} + A_2 e^{-t}) + A_1 e^{-t} + A_2 t e^{-t} + \dot{m} = -A_1 e^{-t} - A_2 t e^{-t} + 2A_2 e^{-t} + \dot{m} = (2A_2 - A_1) e^{-t} - A_2 t e^{-t} + \dot{m} \]

Similar to the expected rate of inflation \( \pi \), the actual rate of inflation \( \dot{p} \) also converges to the intertemporal equilibrium \( \dot{m} \). In fact, in an intertemporal equilibrium, the expected rate of inflation would equal the actual one, so there will be no change in expected inflation. From the first equation of the given model, we can also determine how unemployment changes with time. It is

\[ U = \frac{3 + \pi - \dot{p}}{2} = \frac{\pi - \dot{p}}{2} + \frac{3}{2} \]

Hence, the time path of the rate of unemployment is

\[ U(t) = \frac{1}{2} (A_1 e^{-t} + A_2 t e^{-t} + \dot{m} + A_1 e^{-t} + A_2 t e^{-t} - 2A_2 e^{-t} - \dot{m}) + \frac{3}{2} \text{ or} \]

\[ U(t) = \frac{1}{2} (2A_2 e^{-t} + 2A_2 t e^{-t} - 2A_2 e^{-t}) + \frac{3}{2} = A_1 e^{-t} + A_2 t e^{-t} - A_2 e^{-t} + \frac{3}{2} =
\]

\[ = (A_1 - A_2) e^{-t} + A_2 t e^{-t} + \frac{3}{2} \]

which is also a convergent time path. Expected and real inflation both converge to the intertemporal equilibrium given by the growth rate of nominal money. If this growth rate changes, we will have a moving equilibrium, so the monetary policy of the government will affect the inflation levels. Note, though, that the unemployment rate does not depend on \( \dot{m} \). In this example, it converges to the constant \( \frac{3}{2} \). This constant value is known as the natural rate of unemployment. The independence of unemployment of any equilibrium rate of inflation is reflected by a vertical line known as the long-run Phillips curve. We would normally expect the short-run Phillips curve to be negatively sloped; that is, we would expect higher inflation to be accompanied by a low level of unemployment and vice versa. This short-term negative relationship can easily be deduced from the first equation of the model. The intuitive explanation is that in the business cycle at the time of a boom the economy is nearly at its full-employment level (not accounting for frictional unemployment). At that point, aggregate demand grows substantially exerting an inflationary pressure on prices. Conversely, when in a recession a lot of people are unemployed, aggregate demand falls, pushing prices down to their normal levels.

26. Consider the Phillips relation model

\[ \dot{p} = 5 - 3U + \pi \]

\[ \frac{d\pi}{dt} = \frac{2}{3} (\dot{p} - \pi) \]

\[ \frac{dU}{dt} = -(\dot{m} - \dot{p}) \]

Find the time path of the expected rate of inflation, the actual rate of inflation, and the unemployment rate.
Solution:
For the parameters, we have
\[
\beta = 3 \quad j = \frac{2}{3} \quad h = 1 \quad \text{and} \quad k = 1
\]
For the coefficients \(a_1, a_2\) and \(b\), we have
\[
a_1 = \beta k + j(1 - h) = 3(1) + \frac{2}{3}(0) = 3 \quad a_2 = j\beta k = \frac{2}{3}(3)(1) = 2 \quad b = j\beta km = 2m
\]
The intertemporal equilibrium of the expected rate of inflation is \(\pi_p = \dot{m}\). For the characteristic roots,
\[
r_{1,2} = -a_1 \pm \sqrt{a_1^2 - 4a_2} = \frac{1}{2} \left( -3 \pm \sqrt{9 - 4(2)} \right) = \frac{-3 \pm 1}{2} = -2, -1
\]
which are two distinct roots, so the solution is
\[
\pi(t) = A_1 e^{-t} + A_2 e^{-2t} + \dot{m}
\]
The time path of expected inflation is convergent to the intertemporal equilibrium. Knowing the time path of \(\pi\), we can find that of \(\dot{\pi}\). From the relationship \(\dot{\pi} = \frac{1}{j} \frac{d\pi}{dt} + \pi\), we get
\[
\dot{\pi} = \frac{3}{2} \frac{d\pi}{dt} + \pi
\]
Differentiating \(\pi\),
\[
\pi'(t) = -A_1 e^{-t} - 2A_2 e^{-2t}
\]
and substituting in the expression,
\[
\dot{\pi} = \frac{3}{2}(-A_1 e^{-t} - 2A_2 e^{-2t}) + A_1 e^{-t} + A_2 e^{-2t} + \dot{m} = \frac{1}{2} A_1 e^{-t} - 2A_2 e^{-2t} + \dot{m}
\]
Both the expected rate of inflation \(\pi\) and the actual rate of inflation \(\dot{\pi}\) converge to the intertemporal equilibrium \(\dot{m}\), so they tend to be equal. From the first equation of the given model, we can also determine unemployment. It is
\[
\dot{\pi} = 5 - 3U + \pi
\]
\[
U = \frac{5 + \pi - \dot{\pi}}{3} = \frac{\pi - \dot{\pi} + 5}{3}
\]
Hence, the time path of the rate of unemployment is
\[
U(t) = \frac{1}{3} \left( A_1 e^{-t} + A_2 e^{-2t} + \dot{m} + \frac{1}{2} A_1 e^{-t} + 2A_2 e^{-2t} - \dot{m} \right) + \frac{5}{3} = \frac{1}{2} A_1 e^{-t} + A_2 e^{-2t} + \frac{5}{3}
\]
The unemployment rate converges to the natural rate of unemployment, which in this example happens to be \(\frac{5}{3}\).

27. Let the three equations of the Phillips relation model be
\[
\dot{\pi} = 3 - U + \pi
\]
\[
\frac{d\pi}{dt} = \frac{1}{4} (\dot{\pi} - \pi)
\]
\[ \frac{dU}{dt} = -\frac{1}{2}(\dot{m} - \dot{p}) \]

Find the time path of the expected rate of inflation, the real rate of inflation, and the unemployment rate. Do the functions converge to their equilibrium; and if so, how?

Solution:

We have the following values of the parameters:

\[ \beta = 1 \quad j = \frac{1}{4} \quad h = 1 \quad \text{and} \quad k = \frac{1}{2} \]

Consequently,

\[ a_1 = \beta k + j(1 - h) = \frac{1}{2} \quad a_2 = j \beta k = \frac{1}{8} \quad b = j \beta km = \frac{\dot{m}}{8} \]

Thus, the intertemporal equilibrium is \( \pi_p = \dot{m} \).

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{1}{2} \left( -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{4}{8}} \right) = \frac{1}{2} \left( -\frac{1}{2} \pm \frac{1}{2} \right) = -\frac{1}{4} \pm \frac{1}{4} i \]

The roots are a pair of complex numbers with a real part, \( m = -\frac{1}{4} \), and an imaginary one, \( n = \frac{1}{4} \).

Hence, the general solution for the expected rate of inflation is

\[ \pi(t) = e^{-\frac{t}{4}} \left( B_1 \cos \frac{t}{4} + B_2 \sin \frac{t}{4} \right) + \dot{m} \]

Knowing the time path of \( \pi \), we can find that of \( \dot{p} \). From the relationship \( \dot{p} = \frac{1}{j} \frac{d\pi}{dt} + \pi \), we get

\[ \dot{p} = 4 \frac{d\pi}{dt} + \pi \]

\[ \pi'(t) = -\frac{e^{-\frac{t}{4}}}{4} \left( B_1 \cos \frac{t}{4} + B_2 \sin \frac{t}{4} \right) + e^{-\frac{t}{4}} \left( -\frac{B_1}{4} \sin \frac{t}{4} + \frac{B_2}{4} \cos \frac{t}{4} \right) \]

Substituting for \( \pi \) and \( \pi' \), we obtain for the time path of actual inflation

\[ \dot{p}(t) = e^{-\frac{t}{4}} \left( B_2 \cos \frac{t}{4} - B_1 \sin \frac{t}{4} \right) + \dot{m} \]

And for the unemployment rate,

\[ U = \pi - \dot{p} + 3 \]

\[ U(t) = e^{-\frac{t}{4}} \left( (B_1 - B_2) \cos \frac{t}{4} + (B_1 + B_2) \sin \frac{t}{4} \right) + 3 \]

All the functions fluctuate periodically with a period \( \frac{2\pi}{n} = \frac{2(4\pi)}{1} = 4\pi \). They perform a full cycle every time \( t \) increases by \( 4\pi \). Since the real part is negative or \( m = -\frac{1}{4} \), the fluctuation is damped and the time path is dynamically stable. Both \( \pi \) and \( \dot{p} \) converge cyclically to the intertemporal equilibrium equal to the monetary-policy parameter \( \dot{m} \), the rate of growth of nominal money. This is a moving equilibrium, since this rate of growth would change upon the discretion of the government. The rate of unemployment also fluctuates in a dynamically stable way, but around the natural rate of unemployment, which is equal to 3 here.
28. In the model given in the previous problem, assume that all coefficients are the same, except 
\( h = \frac{1}{3} \). Find \( \pi(t) \), \( p(t) \), and \( U(t) \) and analyze their time paths. Find also the intertemporal equilibria of the variables. What do you notice about the intertemporal equilibrium of the unemployment rate? What can you conclude about the long-term Phillips curve for the new value of the \( h \) parameter?

Solution:

The model now becomes

\[
\dot{p} = 3 - U + \frac{1}{3} \pi \\
\frac{d\pi}{dt} = \frac{1}{4} (\dot{p} - \pi) \\
\frac{dU}{dt} = -\frac{1}{2} (\dot{m} - \dot{p})
\]

Rewriting the parameters,

\[
\beta = 1 \\
j = \frac{1}{4} \\
h = \frac{1}{3} \\
k = \frac{1}{2}
\]

Hence, we obtain

\[
a_1 = \beta k + j(1-h) = \frac{1}{2} + \frac{1}{4} \left(1 - \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\
a_2 = j\beta k = \frac{1}{8} \\
b = j\beta km = \frac{m}{8}
\]

Thus, the intertemporal equilibrium for expected inflation is \( \pi_o = \dot{m} \).

\[
n_{1,2} = -a_1 \pm \sqrt{a_1^2 - 4a_2} \\
= \frac{1}{2} \left(-\frac{2}{3} \pm \frac{4}{9} - \frac{4}{8} \right) \\
= \frac{1}{2} \left(-\frac{2}{3} \pm \frac{\sqrt{2}}{6} i \right) \\
= -\frac{1}{3} \pm \frac{\sqrt{2}}{12} i
\]

These are complex roots with \( m = -\frac{1}{3} \) and \( n = \frac{\sqrt{2}}{12} \). The general solution for the expected rate of inflation is

\[
\pi(t) = e^{-\frac{t}{3}} \left( B_1 \cos \frac{\sqrt{2}t}{12} + B_2 \sin \frac{\sqrt{2}t}{12} \right) + \dot{m}
\]

From the relationship \( \dot{p} = \frac{1}{j} \frac{d\pi}{dt} + \pi \), we have

\[
\dot{p} = 4 \frac{d\pi}{dt} + \pi \\
\pi'(t) = -\frac{e^{-\frac{t}{3}}}{3} \left( B_1 \cos \frac{\sqrt{2}t}{12} + B_2 \sin \frac{\sqrt{2}t}{12} \right) + e^{-\frac{t}{3}} \left( -\frac{\sqrt{2}B_1}{12} \sin \frac{\sqrt{2}t}{12} + \frac{\sqrt{2}B_2}{12} \cos \frac{\sqrt{2}t}{12} \right)
\]

Substituting for \( \pi \) and \( \pi' \), we obtain the time path of actual inflation:

\[
\dot{p}(t) = -\frac{4e^{-\frac{t}{3}}}{3} \left( B_1 \cos \frac{\sqrt{2}t}{12} + B_2 \sin \frac{\sqrt{2}t}{12} \right) + \frac{\sqrt{2}}{3} e^{-\frac{t}{3}} \left( -B_1 \sin \frac{\sqrt{2}t}{12} + B_2 \cos \frac{\sqrt{2}t}{12} \right) + \\
+ e^{-\frac{t}{3}} \left( B_1 \cos \frac{\sqrt{2}t}{12} + B_2 \sin \frac{\sqrt{2}t}{12} \right) + \dot{m}
\]
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\[ \dot{p}(t) = -\frac{e^{\frac{t}{3}}}{3} \left( B_1 \cos \frac{\sqrt{2}t}{12} + B_2 \sin \frac{\sqrt{2}t}{12} \right) + \frac{\sqrt{2}e^{\frac{t}{3}}}{3} \left( -B_1 \sin \frac{\sqrt{2}t}{12} + B_2 \cos \frac{\sqrt{2}t}{12} \right) + \dot{m} \]

\[ \dot{p}(t) = \frac{\sqrt{2} - 1}{3} (B_1 + B_2) e^{\frac{t}{3}} \cos \frac{\sqrt{2}t}{12} - \frac{\sqrt{2} + 1}{3} (B_1 + B_2) e^{\frac{t}{3}} \sin \frac{\sqrt{2}t}{12} + \dot{m} \]

Hence, the equilibrium values of both expected and actual inflation are equal to the nominal money growth \( \dot{m} \). Since both variables are circular functions, they must be fluctuating in time. With the real part of the roots negative we have damped fluctuation, which means they converge to this equilibrium level. For the unemployment rate,

\[ U = \frac{1}{3} \pi - \dot{p} + 3 \]

Substituting the already obtained expressions,

\[ U(t) = \frac{1}{3} e^{\frac{t}{3}} \left( B_1 \cos \frac{\sqrt{2}t}{12} + B_2 \sin \frac{\sqrt{2}t}{12} \right) + \frac{m}{3} - \frac{\sqrt{2} - 1}{3} (B_1 + B_2) e^{\frac{t}{3}} \cos \frac{\sqrt{2}t}{12} + \]

\[ + \frac{\sqrt{2} + 1}{3} (B_1 + B_2) e^{\frac{t}{3}} \sin \frac{\sqrt{2}t}{12} - \dot{m} + 3 \]

\[ U(t) = e^{\frac{t}{3}} \cos \frac{\sqrt{2}t}{12} \left[ B_1 + \frac{\sqrt{2} - 1}{3} (B_1 + B_2) \right] + e^{\frac{t}{3}} \sin \frac{\sqrt{2}t}{12} \left[ \frac{B_2}{3} + \frac{\sqrt{2} + 1}{3} (B_1 + B_2) \right] + \frac{2m}{3} \]

\[ U(t) = \frac{1}{3} e^{\frac{t}{3}} \cos \frac{\sqrt{2}t}{12} \left[ (2 - \sqrt{2}) B_1 - (\sqrt{2} - 1) B_2 \right] + \frac{1}{3} e^{\frac{t}{3}} \sin \frac{\sqrt{2}t}{12} \left[ (\sqrt{2} + 1) B_1 + (2 + \sqrt{2}) B_2 \right] + \frac{2m}{3} \]

Unemployment is also a circular function that fluctuates in a damped way so it converges to its intertemporal equilibrium. Note, though, that now this equilibrium depends on \( \dot{m} \) since it is

\[ \bar{U} = 3 - \frac{2\dot{m}}{3} \]

Hence, the equilibrium level of unemployment depends on the monetary (that is, inflationary) policy of the government and the long-run Phillips curve is no longer vertical. Since \( \bar{p} = \dot{m} \), the inflation rate is negatively related to unemployment in equilibrium. A higher rate of actual inflation is accompanied by a lower unemployment rate. A lower rate of actual inflation may come at the expense of greater unemployment. We obtain this negative correlation between inflation and unemployment in the long run, particularly when the coefficient \( h \) in the first equation of the model is chosen to be less than 1. Therefore, the generally vertical shape of the long-run Phillips curve is contingent on the special value of this parameter. Since here \( h = \frac{1}{3} \), we have a negatively sloped long Phillips curve as the result for the equilibrium unemployment rate show.

29. Solve the following Phillips relation model:

\[ \dot{p} = \frac{1}{5} - U + \frac{1}{4} \pi \]

\[ \frac{d\pi}{dt} = \frac{1}{3} (\dot{p} - \pi) \]

\[ \frac{dU}{dt} = \frac{3}{4} (\dot{m} - \dot{p}) \]

Find \( \pi(t) \), \( p(t) \), and \( U(t) \) and their intertemporal equilibria. What is the slope of the long-term Phillips curve for the specific value of the \( h \) parameter?
Solution:

The parameters are

\[ \beta = 1 \quad j = \frac{1}{3} \quad h = \frac{1}{4} \quad k = \frac{3}{4} \]

For the coefficients \( a_1 \), \( a_2 \) and \( b \),

\[ a_1 = \beta k + j(1-h) = \frac{3}{4} + \frac{1}{3} \left( 1 - \frac{1}{4} \right) = \frac{3}{4} + \frac{1}{4} = 1 \quad a_2 = j \beta k = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4} \quad b = j \beta km = \frac{\dot{m}}{4} \]

The intertemporal equilibrium of the expected rate of inflation is \( \pi_p = \dot{m} \).

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{1}{2} \left( -1 \pm \sqrt{1 - \frac{4}{4}} \right) = -\frac{1}{2} \]

Since this is a single real root, we have

\[ \pi(t) = A_1 e^{-\frac{t}{2}} + A_2 t e^{-\frac{t}{2}} + \dot{m} \]

From the relationship \( \dot{p} = \frac{1}{j} \frac{d\pi}{dt} + \pi \),

\[ \dot{p} = 3 \frac{d\pi}{dt} + \pi \]

Differentiating \( \pi \),

\[ \pi'(t) = -\frac{1}{2} A_1 e^{-\frac{t}{2}} - \frac{1}{2} A_2 t e^{-\frac{t}{2}} + A_2 e^{-t} \]

and substituting in the expression,

\[ \dot{p} = -\frac{3}{2} A_1 e^{-\frac{t}{2}} - \frac{3}{2} A_2 t e^{-\frac{t}{2}} + 3A_1 e^{-\frac{t}{2}} + A_2 e^{-\frac{t}{2}} + A_2 t e^{-\frac{t}{2}} + \dot{m} = \]

\[ = -\frac{1}{2} A_1 e^{-\frac{t}{2}} - \frac{1}{2} A_2 t e^{-\frac{t}{2}} + 3A_1 e^{-\frac{t}{2}} + \dot{m} \]

For unemployment,

\[ U = \pi - 4 \dot{p} + \frac{1}{5} \]

Hence, the time path of the rate of unemployment is

\[ U(t) = \frac{1}{4} \left( A_1 e^{-\frac{t}{2}} + A_2 t e^{-\frac{t}{2}} + \dot{m} + 2A_1 e^{-\frac{t}{2}} + 2A_2 t e^{-\frac{t}{2}} -12A_2 e^{-\frac{t}{2}} - 4\dot{m} \right) + \frac{1}{5} \text{ or} \]

\[ U(t) = \frac{1}{4} \left( 3A_1 e^{-\frac{t}{2}} + 3A_2 t e^{-\frac{t}{2}} - 12A_2 e^{-\frac{t}{2}} - 3\dot{m} \right) + \frac{1}{5} \]

\[ U(t) = \frac{3}{4} A_2 e^{-\frac{t}{2}} + \frac{3}{4} A_2 t e^{-\frac{t}{2}} - 3A_2 e^{-\frac{t}{2}} + \frac{1}{5} - \frac{3\dot{m}}{4} \]

\[ \overline{U} = \frac{1}{5} - \frac{3\dot{m}}{4} \]

Expected and real inflation both converge to the intertemporal equilibrium given by the growth rate of nominal money \( \dot{m} \). The unemployment rate also has a dynamically stable time path, but here the equilibrium is a moving one and depends on government monetary policy. Furthermore, we see that
the intertemporal equilibrium unemployment and inflation are negatively related. Therefore, the long-
run Phillips curve is not vertical, but is negatively sloped with a slope of \(-\frac{3}{4}\). This is because, as we
can notice from the model, \(h = \frac{1}{4}\).

30. Consider the simple inflation-unemployment model

\[
\dot{p} = \alpha - T - \beta U + h\pi, \quad \alpha, \beta > 0, \quad 0 < h \leq 1
\]

\[
\frac{d\pi}{dt} = j(\dot{p} - \pi), \quad 0 < j \leq 1
\]

in which the third equation is dropped as unemployment \(U\) is assumed to be exogenous. What is the
differential equation in \(\pi\) that obtains? Solve it to find the time path of \(\pi\) as well as its intertemporal
equilibrium. How does this new equilibrium differ from the one for actual inflation rate \(\dot{p}\)?

Solution:

Substituting the first equation into the second,

\[
\frac{d\pi}{dt} = j(\alpha - T - \beta U + h\pi - \pi)
\]

\[
\frac{d\pi}{dt} + j(1-h)\pi = j(\alpha - T - \beta U)
\]

which is a first-order differential equation in \(\pi\). It is easy to solve using the definite solution formula

\[
\pi(t) = \left[ \pi(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a} \quad \text{where} \quad a = j(1-h) \quad \text{and} \quad b = j(\alpha - T - \beta U)
\]

Thus, the intertemporal equilibrium of the expected rate of inflation is

\[
\bar{\pi} = \frac{\alpha - T - \beta U}{1-h}
\]

To find the equilibrium value of actual inflation rate, we substitute for \(\bar{\pi}\):

\[
\dot{p} = \alpha - T - \beta U + \frac{h(\alpha - T - \beta U)}{1-h} \quad \text{or}
\]

\[
\bar{p} = \frac{\alpha - T - \beta U}{1-h}
\]

In intertemporal equilibrium, actual and expected inflation rates are equal. This result obtains
alternatively from the second equation of the model by which the change in expected inflation is the
difference between actual and projected inflation. Since in intertemporal equilibrium it could be
expected that \(\bar{\pi}\) will be constant, then \(\bar{\pi} = \bar{p}\). Given that the parameter \(h\) is assumed to be less than
1 (for \(h = 1\) the equilibrium value is undefined), the expected rate of inflation has a dynamically stable
time path and converges to this equilibrium level. Furthermore, since the denominator of \(\bar{p}\) is positive,
if the numerator is positive, then the equilibrium value is positive and there is indeed inflation. If,
however, the numerator turns out to be negative, there is a fall in the average price level – that is,
deflation. For instance, when labor productivity \(T\) is sufficiently high, we may expect a fall in the
price level. Also, in times of a recession or depression we could have a decline in the general price
level again, this time driven by a sizeable level of unemployment due to massive layoffs and limited
aggregate demand.
31. For the general inflation-unemployment model

\[
\dot{p} = \alpha - T - \beta U + h\pi \quad \alpha, \beta > 0 \quad 0 < h \leq 1
\]

\[
\frac{d\pi}{dt} = j(\dot{p} - \pi) \quad 0 < j \leq 1
\]

\[
\frac{dU}{dt} = -k(\dot{m} - \dot{p}) \quad k > 0
\]

we expressed a second-order differential equation in the variable \( \pi \). Write the model alternatively in the form of a second-order differential equation in \( U \). Prove that the coefficients \( a_1 \) and \( a_2 \) are the same but \( b = k\beta [\alpha - T - (1-h)m] \).

**Solution:**

Since both \( \pi \) and \( \dot{p} \) are endogenous and functions of time, in order to solve for \( U(t) \) we need to drop both of these variables from the equation for \( U \). Expressing \( \pi \) from the first equation,

\[
\pi = \frac{\dot{p} - \alpha + T + \beta U}{h}
\]

and substituting it in the second equation,

\[
\frac{d\pi}{dt} = j\left(\dot{p} - \frac{\alpha - T + \beta U}{h}\right) = \frac{j}{h}(h\dot{p} - \dot{p} + \alpha - T - \beta U)
\]

Rewriting the third equation,

\[
\frac{dU}{dt} = -k(\dot{m} - \dot{p}) = -k(\dot{m} - \alpha + T + \beta U - h\pi)
\]

and differentiating both sides with respect to \( t \),

\[
\frac{d^2U}{dt^2} = -\beta k \frac{dU}{dt} + hk \frac{d\pi}{dt}
\]

Substituting for \( \frac{d\pi}{dt} \),

\[
\frac{d^2U}{dt^2} = -\beta k \frac{dU}{dt} + \frac{jkh}{h}(h\dot{p} - \dot{p} + \alpha - T - \beta U)
\]

\[
\frac{d^2U}{dt^2} + \beta k \frac{dU}{dt} = jk(h - 1)\dot{p} + jk(\alpha - T) - jk\beta U
\]

Finally from the last equation of the model, we have for \( \dot{p} \)

\[
\dot{p} = \dot{m} + \frac{1}{k} \frac{dU}{dt}
\]

which we substitute in the second-order differential equation

\[
\frac{d^2U}{dt^2} + \beta k \frac{dU}{dt} = jk(h - 1)\left(\dot{m} + \frac{1}{k} \frac{dU}{dt}\right) + jk(\alpha - T) - jk\beta U
\]

\[
\frac{d^2U}{dt^2} + \left[\beta k + j(1 - h)\right] \frac{dU}{dt} + jk\beta U = jk[\alpha - T - m(1-h)]
\]

Comparing this result to the second-order differential equation for \( \pi(t) \), we see that, indeed, the coefficients \( a_1 \) and \( a_2 \) are the same or \( a_1 = \beta k + j(1-h) \) and \( a_2 = jk\beta \). Just for \( b \) we get
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Thus, the equilibrium value \( \bar{U} \) should be given by the particular integral

\[
U_p = \frac{b}{a_2} = \frac{\alpha - T - m(1 - h)}{\beta}
\]

Solving the second-order differential equation for \( U \) would yield a general solution that should be the same as the time path of \( U \) obtainable through \( \pi \). Furthermore, the intertemporal equilibrium for the unemployment rate should be the same for a particular model.

32. For the model given in problem 28, find the general solution for the time path and equilibrium value of \( U(t) \). Compare the results to the ones obtained previously.

Solution:

The model is

\[
\dot{\pi} = 3 - U + \frac{1}{3} \pi
\]

\[
\frac{d\pi}{dt} = \frac{1}{4} (\dot{\pi} - \pi)
\]

\[
\frac{dU}{dt} = -\frac{1}{2} (\dot{m} - \dot{\pi})
\]

So, for the parameters, we have

\[
\alpha - T = 3 \quad \beta = 1 \quad j = \frac{1}{4} \quad h = \frac{1}{3} \quad \text{and} \quad k = \frac{1}{2}
\]

Hence, we obtain

\[
a_1 = \beta j + j(1 - h) = \frac{1}{2} + \frac{1}{4} \left( 1 - \frac{1}{3} \right) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}
\]

\[
a_2 = j \beta k = \frac{1}{8}
\]

\[
b = jk [\alpha - T - m(1 - h)] = \frac{1}{8} \left[ 3 - \dot{m} \left( 1 - \frac{1}{3} \right) \right] = \frac{1}{8} \left( 3 - \frac{2\dot{m}}{3} \right)
\]

Thus, the intertemporal equilibrium for the rate of unemployment is

\[
U_p = 3 - \frac{2\dot{m}}{3}
\]

which is exactly the intertemporal equilibrium value we obtained previously. Again, since \( h = \frac{1}{3} \), we get a negative relationship between unemployment and equilibrium inflation rate \( \dot{m} \). This results in a negatively sloped long-run Phillips curve. For the characteristic roots, we have

\[
r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{1}{2} \left( -\frac{2}{3} \pm \frac{\sqrt{4 - \frac{4}{9}}}{\frac{4}{3}} \right) = \frac{1}{2} \left( -\frac{2}{3} \pm \frac{\sqrt{2}}{6} i \right) = -\frac{1}{3} \pm \frac{\sqrt{2}}{12} i
\]

These are complex roots with \( m = -\frac{1}{3} \) and \( n = \frac{\sqrt{2}}{12} \). Hence, the general solution for unemployment is

\[
U(t) = e^{-\frac{t}{3}} \left( B_e \cos \frac{\sqrt{2}t}{12} + B_i \sin \frac{\sqrt{2}t}{12} \right) + 3 - \frac{2\dot{m}}{3}
\]

This is quite similar to the function obtained previously, but this time with different coefficients:
33. Consider the extended inflation-unemployment model in which the rate of change of the inflation rate is a decreasing function not only of the level of the unemployment rate, but also of its rate of change (an example of the so-called hysteresis system). Thus, the inflation-unemployment model is

\[
\frac{dp}{dt} = -\alpha(U - U_n) - \delta \frac{dU}{dt}
\]

\[
U = \gamma - \beta \ln \frac{M}{p}
\]

Solve for \( \dot{p} \) and \( U \) and find their equilibrium values.

**Solution:**

Substituting for \( U \),

\[
\frac{d^2 p}{dt^2} = -\alpha \left( \gamma - \beta \ln \frac{M}{p} - U_n \right) - \delta \frac{d}{dt} \left( \gamma - \beta \ln \frac{M}{p} \right)
\]

and differentiating with respect to \( t \),

\[
\frac{d^2 \dot{p}}{dt^2} = \alpha \beta (\dot{m} - \dot{p}) + \beta \delta \frac{d}{dt} (\dot{m} - \dot{p})
\]

\[
\frac{d^2 \dot{p}}{dt^2} = \alpha \beta (\dot{m} - \dot{p}) - \beta \delta \frac{d\dot{p}}{dt}
\]

\[
\frac{d^2 \dot{p}}{dt^2} + \beta \delta \frac{d\dot{p}}{dt} + \alpha \beta \dot{p} = \alpha \beta \dot{m}
\]

Again, nominal money supply \( \dot{m} \) is a stationary value for inflation rate \( \dot{p} \). Here we have \( a_1 = \beta \delta \), \( a_2 = \alpha \beta \), and \( b = \alpha \beta \dot{m} \). Hence, the particular integral is \( \dot{p}_e = \dot{m} \) and the characteristic roots are

\[
r_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4a_2}}{2} = \frac{-\beta \delta \pm \sqrt{\beta^2 \delta^2 - 4\alpha \beta}}{2}
\]

Thus the general solution for inflation would depend on the values of the characteristic roots where if \( \beta \delta^2 \geq 4\alpha \), we have real roots such that

\[
\dot{p}(t) = \dot{m} + A_1 e^{r_1 t} + A_2 e^{r_2 t}
\]

Since the constants \( \beta \delta \) and \( \alpha \beta \) are positive, the roots (or their real part) turn out to be negative, and the equilibrium is dynamically stable. For the unemployment rate, we know that

\[
U = U_n - \frac{1}{\alpha} \frac{d\dot{p}}{dt}
\]

which again gives the natural rate of unemployment as the equilibrium rate for \( U \). The general solution for unemployment by differentiation of the inflation rate

\[
U = U_n - \frac{1}{\alpha} \left( A_r e^{r_1 t} + A_2 r_2 e^{r_2 t} \right)
\]
34. In the extended inflation-unemployment model, assume that the rate of change of the inflation rate is a decreasing function of the level of unemployment but the unemployment rate itself is a decreasing function of both real money supply \( \frac{M}{p} \) and the inflation rate \( \dot{p} \). An increase in \( \dot{p} \), given \( \frac{M}{p} \), increases aggregate demand and, therefore, lowers unemployment. This results in the following inflation-unemployment model:

\[
\frac{d\dot{p}}{dt} = -\alpha(U - U_n) \quad \alpha > 0
\]

\[
U = \gamma - \beta \ln \left( \frac{M}{p} \right) - \sigma \dot{p} \quad \beta, \gamma, \sigma > 0
\]

Solve for \( \dot{p} \) and \( U \) and analyze their time paths.

**Solution:**

Substituting for \( U \),

\[
\frac{d^2 p}{dt^2} = -\alpha \left( \gamma - \beta \ln \left( \frac{M}{p} \right) - \sigma \dot{p} - U_n \right)
\]

and differentiating with respect to \( t \),

\[
\frac{d^2 \dot{p}}{dt^2} = \alpha \beta (\dot{m} - \dot{p}) + \alpha \sigma \frac{dp}{dt}
\]

\[
\frac{d^2 p}{dt^2} = -\alpha \sigma \frac{dp}{dt} + \alpha \beta \dot{p} = \alpha \beta \dot{m}
\]

Again, nominal money supply \( \dot{m} \) is a stationary value for inflation rate \( \dot{p} \). Here we have \( a_1 = -\alpha \sigma \), \( a_2 = \alpha \beta \), and \( b = \alpha \beta \dot{m} \). Hence, the particular integral is \( \dot{p}_p = \dot{m} \) and the characteristic roots are

\[
r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{\alpha \sigma \pm \sqrt{\alpha^2 \sigma^2 - 4 \alpha \beta}}{2}
\]

Thus, the general solution for inflation would depend on the values of the characteristic roots. If it happens that \( \alpha \sigma^2 > 4 \beta \), we have real roots. If \( \alpha \sigma^2 < 4 \beta \), then we get complex roots for the time path of inflation. In all cases, though, we know that this time path is unstable since the parameters \( \alpha \sigma \) and \( \alpha \beta \) are positive and the real part of the characteristic roots is also positive.

\[
\dot{p}(t) = \dot{m} + A_1 e^{r_1 t} + A_2 e^{r_2 t}
\]

From the expression for the unemployment rate, we obtain

\[
U = U_n - \frac{1}{\alpha} \frac{d\dot{p}}{dt}
\]

which again gives the natural rate of unemployment as the equilibrium rate for \( U \). The general solution for unemployment by differentiation of the inflation rate

\[
U = U_n - \frac{1}{\alpha} \left( A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t} \right)
\]

35. The following model is given where the Phillips relation involves aggregate output \( Y \) rather than unemployment. Thus, when actual output exceeds the potential one, actual inflation exceeds the expected one.

\[
\dot{p} = \pi + \alpha(Y - \bar{Y}) \quad \alpha > 0 \quad \text{(Phillips relation)}
\]

\[ \frac{d\pi}{dt} = j(\dot{\pi} - \pi) \quad 0 < j \leq 1 \]  
(adaptive expectations)

\[ \ln \frac{M}{p} = \beta Y - \delta (r + \pi) \quad \beta, \delta > 0 \]  
(LM schedule)

It is still assumed that expectations are adaptive. Finally, the LM schedule gives the relationship between real money supply \( \frac{M}{p} \), where \( M \) is nominal money supply and aggregate output is \( Y \). The government increases money supply as aggregate output expands or as people expect inflation to decline, with \( r \) being the real interest rate. Solve for expected inflation \( \pi \).

**Solution:**

We substitute the first equation into the second:

\[ \frac{d\pi}{dt} = j(\pi + \alpha Y - \alpha \bar{Y} - \pi) \]

\[ \frac{d\pi}{dt} = \alpha j(Y - \bar{Y}) \]

Differentiating with respect to \( t \),

\[ \frac{d^2\pi}{dt^2} = \alpha j \frac{dY}{dt} \]

and differentiating the third equation also with respect to \( t \),

\[ \dot{m} - \dot{p} = \beta \frac{dY}{dt} - \delta \frac{d\pi}{dt} \]

which gives

\[ \frac{dY}{dt} = \frac{1}{\beta} \left( \dot{m} - \dot{p} + \delta \frac{d\pi}{dt} \right) \]

and substituting for \( \frac{dY}{dt} \) in the equation for expected inflation,

\[ \frac{d^2\pi}{dt^2} = \frac{\alpha j}{\beta} \left( \dot{m} - \dot{p} + \delta \frac{d\pi}{dt} \right) \]

And finally, substituting \( \dot{p} = \frac{1}{j} \frac{d\pi}{dt} + \pi \) for \( \dot{p} \) from the last equation,

\[ \frac{d^2\pi}{dt^2} = \frac{\alpha j}{\beta} \left( \dot{m} - \frac{1}{j} \frac{d\pi}{dt} - \pi + \delta \frac{d\pi}{dt} \right) \]

Rearranging,

\[ \frac{d^2\pi}{dt^2} + \alpha j \left( 1 - \delta j \right) \frac{d\pi}{dt} + \frac{\alpha j}{\beta} \pi = \frac{\alpha j}{\beta} \dot{m} \]  
or

\[ \pi'' + \frac{\alpha}{\beta} (1 - \delta j) \pi' + \frac{\alpha j}{\beta} \pi = \frac{\alpha j}{\beta} \dot{m} \]

The parameters in this second-order differential equation are \( a_1 = \frac{\alpha j}{\beta} (1 - \delta j) \), \( a_2 = \frac{\alpha j}{\beta} \), and \( b = \frac{\alpha j}{\beta} \dot{m} \). Thus, \( \pi' = \frac{b}{a_2} = \dot{m} \). The characteristic roots for \( \pi \) are
where the time path of $\pi$ would depend on the particular values of the parameters.

36. For the national-income model in the previous problem, find the intertemporal equilibrium value of aggregate output $Y$ and study its behavior with time. If instead of the absolute value of output, $Y$ denotes the natural log of output, how does the result change for $Y$?

**Solution:**

To trace the time path of aggregate output, we use the equations obtained in the previous example:

$$\frac{dY}{dt} = \frac{1}{\beta} \left( \dot{m} - \dot{p} + \delta \frac{d\pi}{dt} \right)$$

and we differentiate once again with respect to $t$:

$$\frac{d^2Y}{dt^2} = \frac{1}{\beta} \left( \frac{dm}{dt} - \frac{dp}{dt} + \delta \frac{d^2\pi}{dt^2} \right)$$

where $\frac{d^2\pi}{dt^2} = \alpha j \frac{dY}{dt}$

and from $\frac{dp}{dt} = \frac{d\pi}{dt} + \alpha \frac{dY}{dt}$ and $\frac{d\pi}{dt} = \alpha j \left( Y - \bar{Y} \right)$ we have

$$\frac{dp}{dt} = \alpha j \left( Y - \bar{Y} \right) + \alpha \frac{dY}{dt}$$

Substituting in the equation for aggregate output,

$$\frac{d^2Y}{dt^2} = \frac{1}{\beta} \left( \frac{dm}{dt} - \alpha j Y + \alpha j \bar{Y} - \alpha \frac{dY}{dt} + \alpha \delta j \frac{dY}{dt} \right)$$

Rearranging,

$$\frac{d^2Y}{dt^2} + \frac{\alpha (1 - \delta j)}{\beta} \frac{dY}{dt} + \frac{\alpha j}{\beta} Y = \frac{1}{\beta} \left( \dot{m} + \alpha j \bar{Y} \right)$$

where we set $\dot{m}' = \frac{dm}{dt}$

$$Y'' + \frac{\alpha (1 - \delta j)}{\beta} Y' + \frac{\alpha j}{\beta} Y = \frac{1}{\beta} \left( \dot{m}' + \alpha j \bar{Y} \right)$$

Thus,

$$Y_p = \frac{m' + \alpha j \bar{Y}}{\alpha j} = \frac{\dot{m}'}{\alpha j} + \bar{Y}$$

We can conclude that the intertemporal equilibrium value of national income is positively related to the full-employment (potential) output level $\bar{Y}$. Note, however, that the two equilibria are not quite the same. One is a static equilibrium, while the other is a moving equilibrium. Since money growth rate is presumed to change with time, $Y_p$ is a moving equilibrium. The output levels are interrelated, though. Thus, when the government increases nominal money supply at a constant rate $\dot{m}$ (so that $\dot{m}' = 0$), the intertemporal equilibrium is exactly equal to the potential output. Also, intertemporal equilibrium is negatively related to the adjustment coefficient $j$ measuring the discrepancy between...
real and expected inflation as well as $\alpha$ showing the effect an expanding national output has on prices. The differential equation in $Y$ is quite similar to that for expected inflation, which implies that expected inflation and aggregate output have similar time paths. The characteristic roots, hence, are the same as those for expected inflation.

With a log function of aggregate output, the differential equation would be

$$(\ln Y)' + \frac{\alpha(1-\delta j)}{\beta}(\ln Y)' + \frac{\alpha j}{\beta}(\ln Y) = \frac{1}{\beta}(\dot{m}' + \alpha j \ln \bar{Y})$$

where $(\ln Y)' = \dot{Y}$ is the rate of growth of aggregate output and $(\ln Y)'' = \ddot{Y}$ is its rate of change. Hence, the particular integral is

$$(\ln Y)_p = \frac{\dot{m}'}{\alpha j} + \ln \bar{Y}$$

Going a step further and differentiating with respect to $t$, we obtain $\dot{Y}_p = \frac{\dot{m}'}{\alpha j}$. Alternatively, we can solve the differential equation

$$\dot{Y}' + \frac{\alpha(1-\delta j)}{\beta}\dot{Y} + \frac{\alpha j}{\beta}(\ln Y) = \frac{1}{\beta}(\dot{m}' + \alpha j \ln \bar{Y})$$

Differentiating once more transforms the equation into a second-order differential one in the growth rate of national income:

$$\dot{Y}'' + \frac{\alpha(1-\delta j)}{\beta}\dot{Y}' + \frac{\alpha j}{\beta}\dot{Y} = \frac{\dot{m}''}{\beta}$$

Thus, again,

$$\dot{Y}_p = \frac{\dot{m}''}{\alpha j}$$

which shows that at the optimum the government should determine the rate of growth of money supply in pace with the growth rate of equilibrium aggregate output.

37. In the national-income model in problem 35, assume that aggregate output is exogenously determined and find the values of actual and expected inflation.

**Solution:**

The model thus becomes

$$\dot{p} = \pi + \alpha(Y_o - \bar{Y}) \quad \alpha > 0 \quad \text{(Phillips relation)}$$

$$\frac{d\pi}{dt} = j(\dot{p} - \pi) \quad 0 < j \leq 1 \quad \text{(adaptive expectations)}$$

$$\ln \frac{M}{p} = \beta Y_o - \delta(r + \pi) \quad \beta, \delta > 0 \quad \text{(LM schedule)}$$

We substitute the first equation into the second:

$$\frac{d\pi}{dt} = j(\pi + \alpha Y_o - \alpha \bar{Y} - \pi)$$

$$\frac{d\pi}{dt} = \alpha j(Y_o - \bar{Y})$$

From the third equation,
\[ \dot{m} - \dot{p} = -\delta \frac{d\pi}{dt} = -\alpha \delta j(Y_o - \bar{Y}) \]

and hence, for \( \dot{p} \),
\[ \dot{p} = \dot{m} + \alpha \delta j(Y_o - \bar{Y}) \]

From the first equation, we express \( \pi \) :
\[ \pi = \dot{p} - \alpha (Y_o - \bar{Y}) = \dot{m} + \alpha (\delta j - 1)(Y_o - \bar{Y}) \]

We see that in equilibrium, that is, when \( Y_o = \bar{Y} \), both actual and expected inflation would be equal to the growth rate of nominal money supply or
\[ \bar{p} = \bar{\pi} = \dot{m} \]

Furthermore, both types of inflation are nurtured by an aggregate output that grows much above the full-employment level. This causes an overheated economy and an inflationary spiral. A downturn in the economic cycle and a fall of actual output below the normal level leads to low inflation or even deflation.

38. From the national-income model in problem 35, express the intertemporal equilibrium and characteristic roots for actual inflation \( \dot{p} \).

**Solution:**

From the first two equations, we have
\[ \frac{d\pi}{dt} = \alpha j(Y - \bar{Y}) \]

Differentiating the first equation with respect to \( t \),
\[ \frac{dp}{dt} = \frac{d\pi}{dt} + \alpha \frac{dY}{dt} = \alpha j(Y - \bar{Y}) + \alpha \frac{dY}{dt} \]

and differentiating once again,
\[ \frac{d^2\pi}{dt^2} = \alpha j \frac{dY}{dt} + \alpha \frac{d^2Y}{dt^2} \]

From the last equation of the model,
\[ \dot{m} - \dot{p} = \beta \frac{dY}{dt} - \delta \frac{d\pi}{dt} \]

and substituting \( \frac{d\pi}{dt} = \frac{dp}{dt} - \alpha \frac{dY}{dt} \),
\[ \dot{m} - \dot{p} = \beta \frac{dY}{dt} - \delta \left( \frac{dp}{dt} - \alpha \frac{dY}{dt} \right) \]

\[ \dot{m} - \dot{p} = (\beta + \alpha \delta) \frac{dY}{dt} - \delta \frac{dp}{dt} \]

\[ \frac{dY}{dt} = \frac{1}{(\beta + \alpha \delta)} \left( \dot{m} - \dot{p} + \delta \frac{dp}{dt} \right) \]

and differentiating this once again with respect to \( t \),
\[ \frac{d^2Y}{dt^2} = \frac{1}{(\beta + \alpha \delta)} \left( \frac{d\dot{m}}{dt} - \frac{d\dot{p}}{dt} + \delta \frac{d^2p}{dt^2} \right) \]

where we set \( \frac{d\dot{m}}{dt} = \dot{m}' \).
Substituting $\frac{dY}{dt}$ and $\frac{d^2Y}{dt^2}$ in the equation for inflation $\dot{p}$, 

$$\frac{d^2p}{dt^2} = \frac{\alpha j}{(\beta + \alpha j)} \left( m - \dot{p} + \delta \frac{dp}{dt} \right) + \frac{\alpha}{(\beta + \alpha j)} \left( \dot{m}' - \frac{dp}{dt} + \delta \frac{d^2p}{dt^2} \right)$$

$$\frac{d^2p}{dt^2} = \frac{\alpha}{(\beta + \alpha j)} (jm + m') - \frac{\alpha j}{(\beta + \alpha j)} \dot{p} + \frac{\alpha}{(\beta + \alpha j)} (\delta j - 1) \frac{dp}{dt} + \frac{\alpha \delta}{(\beta + \alpha j)} \frac{d^2p}{dt^2}$$

$$(\beta + \alpha j - \alpha \delta) \frac{d^2p}{dt^2} = \alpha (jm + m') - \alpha j \dot{p} + \alpha (\delta j - 1) \frac{dp}{dt}$$

$$(\beta + \alpha j - \alpha \delta) \frac{d^2p}{dt^2} - \alpha (\delta j - 1) \frac{dp}{dt} + \alpha j \dot{p} = \alpha (jm + m')$$

$$\ddot{p} - \frac{\alpha (\delta j - 1)}{\beta + \alpha j - \alpha \delta} \dot{p}' + \frac{\alpha j}{\beta + \alpha j - \alpha \delta} \dot{p} = \frac{\alpha (jm + m')}{\beta + \alpha j - \alpha \delta}$$

Therefore, 

$$\ddot{p} = \frac{\alpha (jm + m')}{\alpha j} = m + \frac{\dot{m}'}{j}$$

In equilibrium $\ddot{p} = \ddot{m} = \ddot{m}'$ such that when money supply is growing at a constant rate, the optimum is $\ddot{p} = \ddot{m} = \ddot{m}'$. The characteristic roots for $\ddot{p}$ are

$$r_{1,2} = \frac{\alpha (\delta j - 1)}{2(\beta + \alpha j - \alpha \delta)} \pm \frac{\sqrt{\alpha^2 (\delta j - 1)^2 - 4 \alpha j}}{2(\beta + \alpha j - \alpha \delta)}$$

$$= \frac{\alpha (\delta j - 1) \pm \sqrt{\alpha^2 (1 - \delta j)^2 - 4 \alpha j (\beta + \alpha j - \alpha \delta)}}{2(\beta + \alpha j - \alpha \delta)} = \frac{\alpha (\delta j - 1) \pm \sqrt{\alpha^2 (1 + \delta j)^2 - 4 \alpha j (\beta + \alpha j)}}{2(\beta + \alpha j - \alpha \delta)}$$

39. When analyzing the quality level $s(t)$ of the items produced in a firm the statisticians found that quality varies according to the differential equation $s'''(t) + 4s''(t) + 6s'(t) + 4s = 0$. If the intertemporal equilibrium represents the desired quality standard or optimal quality level, find that standard and conclude whether, with time, the firm converges to or diverges from this standard.

**Solution:**

This is a third-order differential equation with a particular integral $y_p = \frac{20}{4} = 5$ that is the optimal quality the firm strives to achieve. The characteristic equation is

$$r^3 + 4r^2 + 6r + 4 = 0$$

This transforms into

$$(r + 2)(r^2 + 2r + 2) = 0$$

Thus, we have a real root, $r_1 = -2$, and a pair of complex roots, $r_{2,3} = -1 \pm i$, where $m = -1$ and $n = 1$. The general solution is
\[ s(t) = A_0 e^{-2t} + e^{-t} (B_1 \cos t + B_2 \sin t) + 5 \]

Since both the real root and the real part of the complex roots are negative, the time path of quality is convergent to the equilibrium level of 5. Although the time path is fluctuating around this quality standard, given the circular functions it contains, the firm is reaching the desired level with time. Note that due to the fluctuation, at times the firm might produce below the optimum but also at a quality higher than the required standard. It might be optimal for the firm to produce just at the standard rather than provide too high or too low quality.

40. In studying the dynamics of the value of stock at the stock exchange, the stockbrokers found that the value changes according to the differential equation \[ p^{\prime\prime\prime}(t) - 4 p^{\prime\prime}(t) + 5 p^{\prime}(t) - 2 p = -12 \]. Find the time path of the value \( p(t) \). Is it dynamically stable?

**Solution:**
This is a third-order differential equation with a particular integral \( \bar{p} = \frac{12}{2} = 6 \). The characteristic equation is \( r^3 - 4r^2 + 5r - 2 = 0 \) which can conveniently be transformed into
\[ (r - 2)(r^2 - 2r + 1) = 0 \text{ or} \]
\[ (r - 2)(r - 1)^2 = 0 \]
Thus, we have two real roots, \( r_1 = 1 \) and \( r_2 = 2 \), one of which is repeated. The general solution is
\[ p(t) = A_1 e^t + A_2 t e^t + A_3 e^{2t} + 6 \]
Since both roots are positive and the terms grow infinitely as \( t \to \infty \), the market price of stock is not dynamically stable.

41. Given the multiplier-accelerator model, determine the time path of national income if the accelerator is \( \alpha = 0.8 \) and the marginal propensity to consume is \( \beta = 0.6 \).

**Solution:**
Comparing \( \beta \) and \( \frac{4\alpha}{(1 + \alpha)^2} \) would allow us to determine which particular case we are dealing with.
\[ \frac{4\alpha}{(1 + \alpha)^2} = \frac{4(0.8)}{(1 + 0.8)^2} \approx 0.98 \]
Thus, we have \( \beta < \frac{4\alpha}{(1 + \alpha)^2} \) and \( \alpha \beta < 1 \), which is a time path characterized by damped stepped fluctuation. Thus, the time path of national income, given the values of the parameters, is dynamically stable.

42. In the multiplier-accelerator model, it is given that the accelerator is \( \alpha = 0.3 \), while the simple investment multiplier is 4. Express and analyze the time path of national income.

**Solution:**
We need to find the marginal propensity to consume. From the multiplier-accelerator model, we know that the simple multiplier gives the intertemporal equilibrium of national income
\[ Y_p = \frac{G_o}{1 - \beta} \]

where the multiplier is \( \frac{1}{1 - \beta} = 4 \). We can deduce that \( \beta = 0.75 \). Hence, the intertemporal equilibrium of national income is

\[ Y_p = 4G_o \]

where \( G_o \) is exogenously determined. From the model, we can also express the roots. Comparing \( \beta \) and \( \frac{4\alpha}{(1 + \alpha)^2} \), we establish

\[ \frac{4\alpha}{(1 + \alpha)^2} = \frac{4(3)}{(1 + 3)^2} = 0.75 = \beta \]

Therefore, we have the single real root \( a = \frac{\beta(1 + \alpha)}{2} = \frac{0.75(1 + 3)}{2} = 1.5 \). The function of the national income becomes

\[ Y_t = Y_c + Y_p = A_t Y_c + A_t Y_p = A_t (1.5) + A_t (1.5) + 4G_o \]

Since \( \alpha \beta = 3(0.75) = 2.25 > 1 \) and \( a > 1 \), the time path of national income is nonoscillatory and divergent from the intertemporal equilibrium.

43. If the accelerator is \( \alpha = 0.6 \) and the investment multiplier as defined by Samuelson’s multiplier-accelerator model is 2.5, determine the time path of national income.

**Solution:**

According to the model, the intertemporal equilibrium is given by the particular integral

\[ Y_p = \frac{G_o}{1 - \beta} = 2.5G_o \]

where the multiplier is \( \frac{1}{1 - \beta} = 2.5 \). Thus, we have \( \beta = 0.6 \).

\[ \frac{4\alpha}{(1 + \alpha)^2} = \frac{4(0.6)}{(1 + 0.6)^2} = \frac{2.4}{2.56} = 0.9375 > 0.6 \quad \text{or} \]

\[ \beta < \frac{4\alpha}{(1 + \alpha)^2} \quad \text{and} \quad \alpha \beta < 1 \]

This is the subcase of complex roots where the function of national income demonstrates damped stepped fluctuation to the equilibrium level of \( 2.5G_o \). Alternatively, we could also check that \( R = \sqrt{\alpha \beta} = 0.6 < 1 \), so again the time path is dynamically stable. We can further write the general solution of national income as

\[ Y_c = R^t(B_1 \cos \theta t + B_2 \sin \theta t) = (0.6)^t \left( B_1 \cos \theta t + B_2 \sin \theta t \right) \]

where

\[ \cos \theta = -\frac{b_1}{2\sqrt{b_2}} = \frac{0.6(1.6)}{2(0.6)} = 0.8 \quad \text{and} \quad \sin \theta = \sqrt{1 - \frac{b_1^2}{4b_2}} = \sqrt{1 - 0.8^2} = 0.6 \]

The general solution is

\[ Y_t = (0.6)^t \left( B_1 \cos \theta t + B_2 \sin \theta t \right) + 2.5G_o \]
44. Suppose a particular nation’s marginal propensity to consume is \( \beta = 0.9 \). The nation’s government wants to predict the effect of different accelerator values given by \( \alpha_1 = 0.2 \), \( \alpha_2 = 0.4 \), and \( \alpha_3 = 0.5 \), respectively. Using the framework of the multiplier-accelerator model, help the government analyze the time path of national income. What do you observe about higher values of the accelerator?

**Solution:**

We know that the intertemporal equilibrium of national income is independent of the value of the accelerator, since

\[
Y_p = \frac{G_o}{1 - \beta} = \frac{G_o}{1 - 0.9} = 10G_o
\]

where the multiplier is 10. For \( \alpha_1 = 0.2 \), we have

\[
\frac{4\alpha}{(1 + \alpha)^2} = \frac{4(0.2)}{(1 + 0.2)^2} = \frac{0.8}{1.44} = 0.556 < 0.9 \quad \text{or} \quad \frac{4\alpha}{(1 + \alpha)^2} < \beta
\]

Furthermore, \( \alpha_1\beta = 0.2(0.9) = 0.18 < 1 \). The two results show that the time path of national income is nonoscillatory and convergent (subcase 1 of the distinct-root case). For \( \alpha_2 = 0.4 \), we obtain

\[
\frac{4\alpha}{(1 + \alpha)^2} = \frac{4(0.4)}{(1 + 0.4)^2} = \frac{0.16}{1.96} = 0.816 < 0.9 \quad \text{so again} \quad \frac{4\alpha}{(1 + \alpha)^2} < \beta \quad \text{and} \quad \alpha_2\beta = 0.4(0.9) = 0.36 < 1.
\]

Finally for \( \alpha_3 = 0.5 \),

\[
\frac{4\alpha}{(1 + \alpha)^2} = \frac{4(0.5)}{(1 + 0.5)^2} = \frac{2}{2.25} = 0.889 < 0.9 \quad \text{or} \quad \frac{4\alpha}{(1 + \alpha)^2} < \beta
\]

Again, \( \alpha_3\beta = 0.5(0.9) = 0.45 < 1 \)

All these fall within subcase 1 of the distinct-root case, which means the time path of national income is nonoscillatory and convergent. However, with the increase in the accelerator value we see that there is greater tendency for divergence. It could be checked that for greater values of the accelerator (in any of the three cases) the time path becomes divergent. The simple logic is that when \( \alpha\beta > 1 \), we already have instability. For the particular value of \( \beta = 0.9 \), we obtain that \( 0.9\alpha > 1 \) or \( \alpha > 1.111 \) is a condition for the dynamic instability of national income.

45. For the multiplier-accelerator model, find the characteristic roots and determine the time path of national income if the accelerator is \( \alpha = 3 \) and the marginal propensity to consume is \( \beta = 0.8 \).

**Solution:**

The particular integral gives the intertemporal equilibrium of national income

\[
Y_p = \frac{G_o}{1 - \beta} = \frac{G_o}{1 - 0.8} = 5G_o
\]

From the model, we find the roots

\[
a_{1,2} = \frac{\beta(1 + \alpha) \pm \sqrt{\beta^2(1 + \alpha)^2 - 4\alpha\beta}}{2} = \frac{0.8(1 + 3) \pm \sqrt{0.8^2(1 + 3)^2 - 4(3)0.8}}{2} = \frac{3.2 \pm \sqrt{10.24 - 9.6}}{2} = \frac{3.2 \pm 0.8}{2} = 2; 1.2
\]

The general solution, therefore, is
Comparing $\beta$ and $\frac{4\alpha}{(1+\alpha)^2}$, we find

$$\frac{4\alpha}{(1+\alpha)^2} = \frac{4(3)}{(1+3)^2} = 0.75 < 0.8$$

Since $\alpha\beta = 3(0.8) = 2.4 > 1$ and $\beta > \frac{4\alpha}{(1+\alpha)^2}$, the time path of national income is nonoscillatory and divergent. This is the divergent subcase of the first case of distinct real roots, both of which are greater than one, since $1 < 1.2 < 2$.

### 46. Consider a different form of the multiplier-accelerator model given by

$$Y_t = C_t + I_t$$
$$C_t = \delta + \beta Y_{t-1} \quad 0 < \beta < 1 \quad \delta > 0$$
$$I_t = \gamma + \alpha (Y_{t-1} - Y_{t-2}) \quad \alpha, \gamma > 0$$

where present aggregate investment depends on the increase in national income from the previous period such that the accelerator is still $\alpha$. Solve for national income.

**Solution:**

Substituting the last two equations into the first, we obtain

$$Y_t = \delta + \beta Y_{t-1} + \gamma + \alpha Y_{t-1} - \alpha Y_{t-2}$$
$$Y_t - (\alpha + \beta)Y_{t-1} + \alpha Y_{t-2} = \gamma + \delta$$

Extrapolating this equation by two time periods gives the more convenient form

$$Y_{t+2} - (\alpha + \beta)Y_{t+1} + \alpha Y_t = \gamma + \delta$$

We have $b_1 = -(\alpha + \beta)$, $b_2 = \alpha$, and $c = \gamma + \delta$. The intertemporal equilibrium national income is

$$Y_p = \frac{\gamma + \delta}{1+b_1+b_2} = \frac{\gamma + \delta}{1-\alpha - \beta + \alpha} = \frac{\gamma + \delta}{1-\beta}$$

Note that again $\frac{1}{1-\beta}$ is the value of the multiplier. Furthermore, $\gamma$ and $\delta$, being autonomous consumption and autonomous investment, respectively, influence equilibrium national income positively. The larger the two types of autonomous spending, the greater the value of the multiplier is.

For the characteristic roots, we obtain $a_1 + a_2 = \alpha + \beta$ and $a_1a_2 = \alpha$

$$(1-a_1)(1-a_2) = 1 - (a_1 + a_2) + a_1a_2 = 1 - \alpha - \beta + \alpha = 1 - \beta$$
$$0 < (1-a_1)(1-a_2) < 1$$

Similar to the standard model, we could have several possibilities where the condition for dynamic stability again is for both roots to be fractions. Then $a_1a_2 = \alpha < 1$, or, the requirement is for the accelerator to be less than 1. In the case of repeated and complex roots, we get that $\sqrt{\alpha} < 1$ which is the same condition.

### 47. Given the specific multiplier-accelerator model

$$Y_t = C_t + I_t$$
where the accelerator is $1/4$, solve for national income finding its intertemporal equilibrium and time path. Analyze the behavior of national income in time.

**Solution:**

Substituting into the first equation,

$$Y_t = 10 + \frac{3}{4} Y_{t-1} + 5 + \frac{1}{4} Y_{t-1} - \frac{1}{4} Y_{t-2}$$

$$Y_t - Y_{t-1} + \frac{1}{4} Y_{t-2} = 15$$

Transforming into a more convenient form,

$$Y_{t+2} - Y_{t+1} + \frac{1}{4} Y_{t} = 15$$

Here we have $b_1 = -1$, $b_2 = \frac{1}{4}$ and $c = 15$. Thus, the intertemporal equilibrium national income is

$$Y_p = \overline{Y} = \frac{c}{1+b_1+b_2} = \frac{15}{1-1+\frac{1}{4}} = 60$$

Solving for the characteristic roots,

$$a_{1,2} = -\frac{b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} = \frac{1 \pm \sqrt{1 - \frac{4(1)}{4}}}{2} = \frac{1}{2}$$

which is a case of a real, single root. Hence, the general solution for the time path of national income, given the assumptions of the model, is

$$Y_t = Y_c + Y_p = A_1(0.5)^t + A_2t(0.5)^t + 60$$

Since the root is less than 1, we conclude that the time path of national income is dynamically stable; that is, as $t \to \infty$, national income would tend to approach the equilibrium value of 60.

48. Solve the standard Phillips curve model for expected inflation $\pi$. Assume that

$$\dot{p}_t = \alpha - \beta U_t + h \pi_t$$

$$\pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t)$$

$$U_{t+1} - U_t = -k(m - \dot{p}_{t+1})$$

**Solution:**

This time, we use the difference for $\pi_t$ and substitute for $p_t$:

$$\pi_{t+1} - \pi_t = j(\alpha - \beta U_t + h \pi_t - \pi_t)$$

$$\pi_{t+1} - (1 - j + jh)\pi_t = j\alpha - j \beta U_t$$

Extending this by one time period,

$$\pi_{t+2} - (1 - j + jh)\pi_{t+1} = j\alpha - j \beta U_{t+1}$$

Subtracting the last two equations gives a difference term for unemployment $U$:
\[ \pi_{t+2} - (2 - j + jh)\pi_{t+1} + (1 - j + jh)\pi_t = -j\beta(U_{t+1} - U_t) \]
\[ \pi_{t+2} - (2 - j + jh)\pi_{t+1} + (1 - j + jh)\pi_t = j\beta k (\dot{m} - \dot{\pi}_{t+1}) \]

But from the second equation of the model, we also have

\[ j\dot{\pi}_{t+1} = \pi_{t+2} - (1 - j)\pi_{t+1} \]

And substituting this expression in the equation finally gives a second-order difference equation solely in \( \pi \):

\[ \pi_{t+2} - (2 - j + jh)\pi_{t+1} + (1 - j + jh)\pi_t = j\beta km - \beta k \pi_{t+2} + \beta k(1 - j)\pi_{t+1} \]

\[ (1 + \beta k)\pi_{t+2} - [1 + jh + (1 - j)(1 + \beta k)]\pi_{t+1} + (1 - j + jh)\pi_t = j\beta km \] or

\[ \pi_{t+2} - \frac{[1 + jh + (1 - j)(1 + \beta k)]\pi_{t+1} + (1 - j + jh)\pi_t}{1 + \beta k} = \frac{j\beta km}{1 + \beta k} \]

But this equation is absolutely identical to the one for actual inflation \( \dot{p} \).

\[ (1 + \beta k)\dot{p}_{t+2} - (1 - j + hj + 1 + \beta k)\dot{p}_{t+1} + (1 - j + hj)\dot{p}_t + j\beta(U_{t+1} - U_t) = 0 \]

Hence, the equilibrium value for expected inflation is \( \pi = \frac{c}{1 + a_1 + a_2} = \dot{m} \), and all other conclusions relevant to actual inflation relate also to expected inflation.

49. Consider the standard inflation-unemployment model in discrete time. Assume that the change in unemployment depends on inflation rate from the previous period such that \( U_{t+1} - U_t = -k(\dot{m} - \dot{\pi}_t) \).

Express equilibrium inflation rate.

**Solution:**

The model becomes

\[ \dot{p}_t = \alpha - \beta U_t + h\pi_t \] \( \alpha, \beta > 0 \) \( 0 < h \leq 1 \)
\[ \pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) \] \( 0 < j \leq 1 \)
\[ U_{t+1} - U_t = -k(\dot{m} - \dot{p}_t) \] \( k > 0 \)

Finding the difference for actual inflation,

\[ \Delta p_t = p_{t+1} - p_t \] \( \text{where } \dot{p}_{t+1} = \alpha - \beta U_{t+1} + h\pi_{t+1} \)

\[ \dot{p}_{t+1} - \dot{p}_t = -\beta(U_{t+1} - U_t) + h(\pi_{t+1} - \pi_t) \]

and substituting the last two equations of the model into this expression,

\[ \dot{p}_{t+1} - \dot{p}_t = \beta k (\dot{m} - \dot{p}_t) + hj(\dot{p}_t - \pi_t) \]

\[ \dot{p}_{t+1} - (1 - \beta k + jh)\dot{p}_t = \beta km - hj\pi_t \] \( \text{where } h\pi_t = \dot{p}_t - \alpha + \beta U_t \)

and substituting further,

\[ \dot{p}_{t+1} - (1 - \beta k + jh)\dot{p}_t = \beta km - j\dot{p}_t + j\alpha - j\beta U_t \] \( \text{or} \)
\[ \dot{p}_{t+1} - (1 - \beta k + jh - j)\dot{p}_t = \beta km + j\alpha - j\beta U_t \]

and extending this by one time period,

\[ \dot{p}_{t+2} - (1 - \beta k + jh - j)\dot{p}_{t+1} = \beta km + j\alpha - j\beta U_{t+1} \]
Subtracting the last two equations and substituting for the difference term in inflation finally gives
\[\dot{p}_{t+2} - (2 - \beta k + jh - j) \dot{p}_{t+1} + (1 - \beta k + jh - j) p_t = -j \beta k (U_{t+1} - U_t)\]
\[\dot{p}_{t+2} - (2 - \beta k + jh - j) \dot{p}_{t+1} + (1 - \beta k + jh - j) p_t = j \beta k (\dot{m} - \dot{p}_t)\]
\[\dot{p}_{t+2} - (2 - \beta k + jh - j) \dot{p}_{t+1} + (1 - \beta k + jh - j + j \beta k) p_t = j \beta k \dot{m}\]

For the equilibrium actual inflation rate, we get
\[\bar{p} = \frac{c}{1 + b_1 + b_2} = \frac{j \beta k \dot{m}}{1 - 2 + \beta k - jh + 1 - \beta k + jh - j + j \beta k}\]

This result is consistent with the previous findings of the model according to which the rate of growth of nominal money supply gives the intertemporal equilibrium for actual inflation rate. Note that this result obtains whether the increase in unemployment is assumed to depend on inflation in the current or previous periods.

50. Assume that the inflation-unemployment model is
\[\dot{p}_t = \alpha - \beta U_t + h \pi_t\]
\[\pi_{t+1} - \pi_t = j (\dot{p}_t - \pi_t)\]
\[U_{t+1} - U_t = -k (\dot{m} - \dot{p}_{t+1})\]
so that actual inflation in the present period depends on people’s expectations of inflation in the previous period. Write the difference equation for actual inflation $\dot{p}_t$. What is the order of the equation that obtains? Using the particular integral, find the intertemporal equilibrium of actual inflation rate.

**Solution:**
Alternatively, the model can be written in the form
\[\dot{p}_{t+1} = \alpha - \beta U_{t+1} + h \pi_{t+1}\]
\[\pi_{t+2} - \pi_{t+1} = j (\dot{p}_{t+1} - \pi_{t+1})\]
\[U_{t+2} - U_{t+1} = -k (\dot{m} - \dot{p}_{t+2})\]

Extending the equation for actual inflation by one more period,
\[\dot{p}_{t+2} = \alpha - \beta U_{t+2} + h \pi_{t+2}\]
\[\dot{p}_{t+1} = \alpha - \beta U_{t+1} + h \pi_{t+1}\]
and expressing the difference,
\[\dot{p}_{t+2} - \dot{p}_{t+1} = -\beta (U_{t+2} - U_{t+1}) + h (\pi_{t+2} - \pi_{t+1})\]

and substituting the last two equations of the model into this expression,
\[\dot{p}_{t+2} - \dot{p}_{t+1} = \beta k (\dot{m} - \dot{p}_{t+2}) + h j (\dot{p}_t - \pi_t)\]

Expressing $\pi_t = \frac{\pi_{t+1} - \alpha + \beta U_{t+1}}{h}$ from the first equation,
\[\dot{p}_{t+2} - \dot{p}_{t+1} = \beta k (\dot{m} - \dot{p}_{t+2}) + j h (\dot{p}_t - \dot{p}_{t+1} + \alpha - \beta U_{t+1})\]
\[(1 + \beta k) \dot{p}_{t+2} - (1 - j) \dot{p}_{t+1} - j h \dot{p}_t = \beta k \dot{m} + \alpha j - \beta j U_{t+1}\]
\[(1 + \beta k) \dot{p}_{t+2} - (1 - j) \dot{p}_{t+1} - j h \dot{p}_{t+1} = \beta k \dot{m} + \alpha j - \beta j U_t\]

and subtracting both sides of the last two equations to find the difference again,
\[(1 + \beta k) \dot{p}_{t+2} - (1 - j + 1 + \beta k) \dot{p}_{t+1} - (j h - 1 + j) \dot{p}_t + j h \dot{p}_{t+1} = -\beta j (U_{t+1} - U_t)\]

Substituting for the difference of unemployment,
Normalizing the equation and extending it by one time period finally gives
\[
\dot{p}_{t+3} - \frac{(2-j+\beta k-j\beta k)}{1+\beta k} \dot{p}_{t+2} - \frac{(j\beta k-1+\beta k)}{1+\beta k} \dot{p}_{t+1} + \frac{j\beta k m}{1+\beta k} = \dot{m}
\]
We see that this is a third-order difference equation in \( \dot{p} \), which could be solved given specific values of the parameters. Using the steps for finding the particular integral of such a third-order difference equation, we get
\[
\tilde{p} = \frac{c}{1+b_1+b_2+b_3} = \frac{j\beta k m}{(1+\beta k)(1+\beta k-2+j-\beta k+\beta j k-jh+1-j-jh)} = \dot{m}
\]
Our findings again are consistent with the standard inflation-unemployment model where the rate of growth of nominal money supply gives the intertemporal equilibrium of actual inflation rate. This result obtains whether actual inflation is assumed to depend on past or on present expectations.

51. For the model in the previous problem, assume that the change in unemployment results from inflation in the previous, not in the current, period, such that \( U_{t+1} - U_t = -k(\dot{m} - \dot{p}) \). Write again the difference equation for actual inflation \( \dot{p} \). What is the order of the equation that obtains? Given this new assumption, express intertemporal equilibrium actual inflation rate.

**Solution:**
Alternatively, the model can be written in the form
\[
\begin{align*}
\dot{p}_{t+1} &= \alpha - \beta U_{t+1} + h\pi_t \\
\pi_{t+2} - \pi_{t+1} &= j(\dot{p}_{t+1} - \pi_{t+1}) \\
U_{t+2} - U_{t+1} &= -k(\dot{m} - \dot{p}_{t+1})
\end{align*}
\]
Extending the equation for actual inflation by one more period,
\[
\begin{align*}
\dot{p}_{t+2} &= \alpha - \beta U_{t+2} + h\pi_{t+1} \\
\dot{p}_{t+1} &= \alpha - \beta U_{t+1} + h\pi_t \\
\end{align*}
\]
and expressing the difference,
\[
\dot{p}_{t+2} - \dot{p}_{t+1} = -\beta(U_{t+2} - U_{t+1}) + h(\pi_{t+1} - \pi_t)
\]
and substituting the last two equations of the model into this expression,
\[
\begin{align*}
\dot{p}_{t+2} - \dot{p}_{t+1} &= \beta k(\dot{m} - \dot{p}_{t+1}) + j(\dot{p}_t - \pi_t) \\
where \quad \pi_t &= \frac{\pi_{t+1} - \alpha + \beta U_{t+1}}{h} \\
\dot{p}_{t+2} - \dot{p}_{t+1} &= \beta k(\dot{m} - \dot{p}_{t+1}) + j(\dot{p}_t - \pi_t + \alpha - \beta U_{t+1}) \\
\dot{p}_{t+2} - (1-\beta k-j)\dot{p}_{t+1} - jh\dot{p}_t &= \beta k m + \alpha j - \beta j U_{t+1} \\
\dot{p}_{t+1} - (1-\beta k-j)\dot{p}_t - jh\dot{p}_{t-1} &= \beta k m + \alpha j - \beta j U_t \\
Subtracting,
\dot{p}_{t+2} - (2-\beta k-j)\dot{p}_{t+1} - (jh-1+\beta k+j)\dot{p}_t + jh\dot{p}_{t-1} &= -\beta j(U_{t+1} - U_t) \\
\dot{p}_{t+2} - (2-\beta k-j)\dot{p}_{t+1} - (jh-1+\beta k+j)\dot{p}_t + jh\dot{p}_{t-1} &= j\beta k(\dot{m} - \dot{p}_t) \\
\dot{p}_{t+2} - (2-\beta k-j)\dot{p}_{t+1} - (jh-1+\beta k+j)\dot{p}_t + jh\dot{p}_{t-1} &= j\beta k m
Extending the equation by one time period,
\[ \dot{p}_{t+3} - (2 - \beta k - j) \dot{p}_{t+2} - (j h - 1 + \beta k + j - j \beta k) \dot{p}_{t+1} + j h \dot{p}_t = j \beta km \]
Again, we have a third-order difference equation in \( \dot{p} \), which could be solved following the steps of higher-order difference equations. For the equilibrium value of inflation, we use the particular integral
\[ \ddot{p} = \frac{c}{1 + b_1 + b_2 + b_3} = \frac{j \beta km}{1 - 2 + \beta k + j - jh + 1 - \beta k - j + j \beta k + jh} = m \]
The result again is consistent with the one obtained previously. Whether actual inflation is assumed to depend on past or present expectations, or unemployment depends on previous or current inflation, the equilibrium value for actual inflation does not change.

52. Given the third-order difference equation for \( \dot{p} \) in the previous problem, assume the equation takes the specific form \( \dot{p}_{t+3} - \frac{1}{3} \dot{p}_{t+2} - \frac{1}{4} \dot{p}_{t+1} + \frac{1}{12} \dot{p}_t = 16 \). Using the steps of third-order difference equations, find the general solution for inflation \( \dot{p} \). What is its intertemporal equilibrium, and does inflation converge to or diverge from it?

Solution:
Since we have \( b_1 = -\frac{1}{3} \), \( b_2 = -\frac{1}{4} \) and \( b_3 = \frac{1}{12} \), the particular integral is
\[ \ddot{p} = \frac{c}{1 + b_1 + b_2 + b_3} = \frac{16}{1 - \frac{1}{3} - \frac{1}{4} + \frac{1}{12}} = \frac{16(12)}{12 - 4 - 3 + 1} = 32 \]
The characteristic equation is
\[ a^3 - \frac{1}{3} a^2 - \frac{1}{4} a + \frac{1}{12} = 0 \]
Factoring out the term \( a^2 - \frac{1}{4} \) gives
\[ a \left( a^2 - \frac{1}{4} \right) - \frac{1}{3} \left( a^2 - \frac{1}{4} \right) = 0 \quad \text{or} \]
\[ a - \frac{1}{2} \left( a + \frac{1}{2} \right) \left( a - \frac{1}{3} \right) = 0 \]
Thus, the characteristic roots are \( a_1 = -\frac{1}{2} \), \( a_2 = \frac{1}{2} \), and \( a_3 = \frac{1}{3} \). The general solution of actual inflation, therefore, can be written as
\[ p_t = A_1 \left( -\frac{1}{2} \right)^t + A_2 \left( \frac{1}{2} \right)^t + A_3 \left( \frac{1}{3} \right)^t + 32 \]
We can further definitize the arbitrary constants, if we are given some initial conditions. Since all the three roots have absolute values smaller than 1, the time path of inflation is convergent to the intertemporal equilibrium of 32.

53. Consider a simplified inflation-unemployment model where the unemployment rate is assumed to be exogenous:
\[ \dot{p}_t = \alpha - \beta U_o + h \pi_t \quad \alpha, \beta > 0 \quad 0 < h \leq 1 \]
\[ \pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) \quad 0 < j \leq 1 \]

Find and determine the time path of actual inflation rate \( \dot{p} \). What is the intertemporal equilibrium value of inflation? What kind of difference equation obtains?

**Solution:**

Using the difference

\[ \Delta p_t = p_{t+1} - p_t \]

where we have \( \dot{p}_{t+1} = \alpha - \beta U_o + h\pi_{t+1} \)

\[ \dot{p}_{t+1} - \dot{p}_t = h(\pi_{t+1} - \pi_t) \]

and substituting the difference for expected inflation,

\[ \dot{p}_{t+1} - \dot{p}_t = hj(\dot{p}_t - \pi_t) \]

From the first equation of the model, we also have \( h\pi_t = \dot{p}_t - \alpha + \beta U_o \), and substituting further,

\[ \dot{p}_{t+1} - \dot{p}_t = hj\dot{p}_t - j\dot{p}_t + j(\alpha - \beta U_o) \]

\[ \dot{p}_{t+1} - (1 - j + jh)\dot{p}_t = j(\alpha - \beta U_o) \]

which is a first-order difference equation solely in \( \dot{p} \). We know from before that the general solution for the time path of actual inflation can by found by the formula

\[ \dot{p}_t = \left( \dot{p}_o - \frac{c}{1+b} \right)(-b)^t + \frac{c}{1+b} \quad \text{where} \quad b = -(1 - j + jh) \quad \text{and} \quad c = j(\alpha - \beta U_o) \]

Substituting,

\[ \dot{p}_t = \left[ \dot{p}_o - \frac{j(\alpha - \beta U_o)}{1 - 1 + j - jh} \right](1 - j + jh)^t + \frac{j(\alpha - \beta U_o)}{1 - 1 + j - jh} \]

\[ \dot{p}_t = \left[ \dot{p}_o - \frac{(\alpha - \beta U_o)}{1 - h} \right](1 - j + jh)^t + \frac{(\alpha - \beta U_o)}{1 - h} \]

where the equilibrium value for the inflation rate is \( \pi^* = \frac{\alpha - \beta U_o}{1 - h} \). Although the result is different from the one for the expanded model, we still get an inverse relationship between inflation and unemployment, which illustrates the negatively sloped long-run Phillips curve. Analyzing the time path further, we find out that the term \( 1 - j + jh \) is always less than 1, since \( 0 < j, h \leq 1 \). Therefore, the time path is convergent. Since this term is also positive, it follows that the time path is nonoscillatory.

54. For the simplified model in the previous problem, obtain the time path of expected inflation \( \pi \). How does it differ from the one for real inflation \( \dot{p} \)?

**Solution:**

We can solve easily taking the difference for expected inflation from the second equation

\[ \pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) \]

and substituting the term \( \dot{p}_t \),

\[ \pi_{t+1} - \pi_t = j(\alpha - \beta U_o + h\pi_t - \pi_t) \]
\[ \pi_{t+1} - (1 + jh - j)\pi_t = j(\alpha - \beta U_o) \]

which is again a first-order difference equation in \( \pi \). Thus, the general solution is

\[
\pi_t = \left[ \pi_o - \frac{(\alpha - \beta U_o)}{1-h} \right] (1 - j + jh)^t + \frac{(\alpha - \beta U_o)}{1-h}
\]

where the equilibrium expected inflation rate is \( \bar{\pi} = \frac{\alpha - \beta U_o}{1-h} \). This is the same as the value obtained for actual inflation. Logically, in a state of equilibrium the value of the two inflation rates should be equal. Similar to actual inflation, expected inflation is convergent and nonoscillatory.

55. Consider the simplified inflation-unemployment model in which unemployment is assumed to be exogenous. Imagine that there is a time lag in the way people’s inflationary expectations form actual inflation. In other words, actual inflation in the present period depends on expected inflation from the previous period. Hence, the model is

\[
\dot{p}_{t+1} = \alpha - \beta U_o + h\pi_t, \quad \alpha, \beta > 0 \quad 0 < h \leq 1
\]

Solve for actual inflation rate \( \dot{p} \). What is its intertemporal equilibrium value, and how does it depend on expectations? Analyze also its time path.

Solution:

\[
\Delta p_t = p_{t+2} - p_{t+1} \quad \text{where we have} \quad \dot{p}_{t+1} = \alpha - \beta U_o + h\pi_t \quad \text{and} \quad \dot{p}_{t+2} = \alpha - \beta U_o + h\pi_{t+1}
\]

\[
\dot{p}_{t+2} - \dot{p}_{t+1} = h(\pi_{t+1} - \pi_t) = jh(\dot{p}_t - \pi_t)
\]

From the first equation of the model, we also have \( h\pi_t = \dot{p}_{t+1} - \alpha + \beta U_o \), and substituting further,

\[
\dot{p}_{t+2} - \dot{p}_{t+1} = hj\dot{p}_t - j\dot{p}_{t+1} + j(\alpha - \beta U_o)
\]

\[
\dot{p}_{t+2} - (1 - j)\dot{p}_{t+1} - hj\dot{p}_t = j(\alpha - \beta U_o)
\]

This is a second-order difference equation where the equilibrium value for the inflation rate is

\[
\bar{p} = \frac{c}{1+b_1+b_2} = \frac{\alpha - \beta U_o}{1-h}
\]

This is a result we obtained previously with unemployment again considered exogenous. It shows that equilibrium actual inflation is unaffected by people’s expectations. Whether those expectations were formed in the previous or the current period, the equilibrium level of inflation stays the same.

\[
a_{t,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} = \frac{1 - j \pm \sqrt{(1-j)^2 - 4jh}}{2}
\]

and we have different outcomes depending on whether \((1-j)^2 > 4jh\). We also know that the characteristic roots must satisfy the conditions \( a_1 + a_2 = -b_1 = 1 - j \) and \( a_1a_2 = b_2 = -jh \). Then it must be that

\[
(1-a_1)(1-a_2) = 1 - (a_1 + a_2) + a_1a_2 = 1 - 1 - j - jh = j(1-h) > 0
\]

Since \( 0 < j, h \leq 1 \), we conclude that

\[
a_1 + a_2 > 0 \quad a_1a_2 < 0 \quad \text{and} \quad (1-a_1)(1-a_2) > 0
\]
so one of the roots \( a_2 \) is positive and the other \( a_1 \) is negative where the two must be fractions, but \( a_2 \) prevails over \( a_1 \); that is, \( 1 > |a_2| > |a_1| \). Since the two roots are fractions smaller than 1, the time path of actual inflation \( \dot{p} \) must be convergent.

**56.** Consider the extended inflation-unemployment model, dealt with previously, in its continuous-time form

\[
\frac{dp}{dt} = -\alpha(U - U_n) \quad \alpha > 0 \\
\frac{dU}{dt} = -\beta(m - \dot{p}) \quad \beta > 0
\]

where \( U \) is the rate of actual unemployment while \( U_n \) is a fixed, natural rate of unemployment. Convert the model in a discrete-time form and solve for the time path of inflation \( \dot{p} \).

**Solution:**

From the first equation of the model, by further differentiation we obtained

\[
\frac{d^2\dot{p}}{dt^2} = -\alpha \frac{dU}{dt}
\]

In discrete time this should involve a second difference of price on the left side, or

\[
\Delta^2 \dot{p}_t = \Delta(\Delta\dot{p}_t) = \Delta(\dot{p}_{t+1} - \dot{p}_t) = (\dot{p}_{t+2} - \dot{p}_{t+1}) - (\dot{p}_{t+1} - \dot{p}_t) = \dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t
\]

The equation in its discrete form becomes

\[
\dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = -\alpha(U_{t+1} - U_t)
\]

where from the second equation of the model we have in discrete time

\[
U_{t+1} - U_t = -\beta(m - \dot{p}_t)
\]

Thus, the new model becomes

\[
\dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = -\alpha(U_{t+1} - U_t)
\]

where \( U_{t+1} - U_t = -\beta(m - \dot{p}_t) \)

Substituting the difference term for unemployment gives a second-order difference equation in \( \dot{p} \):

\[
\dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = \alpha\beta(m - \dot{p}_t) \quad \text{or}
\]

\[
\dot{p}_{t+2} - 2\dot{p}_{t+1} + (1 + \alpha\beta)\dot{p}_t = \alpha\beta m
\]

The equilibrium value for \( \dot{p} \) is \( \bar{p} = \frac{\alpha\beta m}{1 - 2 + 1 + \alpha\beta} = \bar{m} \). This result is consistent with our previous findings. For the characteristic roots, we get

\[
a_{1,2} = -b_1 \pm \sqrt{b_1^2 - 4b_2} = 2 \pm \sqrt{4 - 4(1 + \alpha\beta)} = 2 \pm 2i\sqrt{\alpha\beta} = 1 \pm i\sqrt{\alpha\beta}
\]

which turn out to be complex numbers, so the time path of the inflation rate must involve stepped fluctuation. Since \( R = \sqrt{b_2} = \sqrt{(1 + \alpha\beta)} \) where both \( \alpha \) and \( \beta \) are positive constants, it must be that \( R > 1 \). Hence, the fluctuating path of inflation, given the assumptions of the model, must be explosive.
57. For the discrete time model in the previous problem, assume that the difference for unemployment is given by \( U_{t+1} - U_t = -\beta (\dot{m} - \dot{p}_{t+1}) \), that is the increase in unemployment depends on inflation in the present, not in the previous period.

**Solution:**

In this new version, the model becomes

\[
p_{t+2} - 2p_{t+1} + \dot{p}_t = -\alpha (U_{t+1} - U_t)
\]

\[
U_{t+1} - U_t = -\beta (\dot{m} - \dot{p}_{t+1})
\]

Substituting again, the difference term for unemployment results in

\[
p_{t+2} - 2p_{t+1} + \dot{p}_t = \alpha \beta (\dot{m} - \dot{p}_{t+1})
\]

\[
\dot{p}_{t+2} - (2 - \alpha \beta) \dot{p}_{t+1} + \dot{p}_t = \alpha \beta \dot{m}
\]

The equilibrium value for \( \dot{p} \) is \( \dot{p} = \frac{\alpha \beta \dot{m}}{1 - \alpha \beta} \). Again, the intertemporal equilibrium of inflation is the growth rate of nominal money supply. The characteristic roots are

\[
a_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} = \frac{2 - \alpha \beta \pm \sqrt{(2 - \alpha \beta)^2 - 4}}{2} = \frac{2 - \alpha \beta \pm \sqrt{\alpha \beta (\alpha \beta - 4)}}{2}
\]

By analyzing the roots further, we find

\[
a_1 + a_2 = -b_1 = 2 - \alpha \beta \quad \text{and} \quad a_1 a_2 = b_2 = 1.
\]

\[
(1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1 a_2 = 1 - 2 + \alpha \beta + 1 = \alpha \beta > 0
\]

Since both \( \alpha \) and \( \beta \) are positive constants, one possibility is for both roots to be negative where one is a fraction. (Can you see why the two roots cannot both be fractions?) From the second equation, we also see that one root is reciprocal of the other. Therefore, we conclude that

\[
a_1, a_2 < 0 \quad \text{and} \quad |a_1| > 1 \quad \text{and} \quad |a_2| < 1
\]

Since the absolute value of one of the roots turns out to be greater than 1, the time path of inflation is divergent and nonoscillatory.

58. The extended inflation-unemployment model in its continuous-time form is

\[
\frac{d\dot{p}}{dt} = -\alpha (U - U_n) - \delta \frac{dU}{dt}
\]

\[
\frac{dU}{dt} = -\beta (\dot{m} - \dot{p})
\]

where \( U \) is the rate of actual unemployment and \( U_n \) is the natural rate of unemployment. Convert the model in a discrete-time form and solve for the time path of inflation \( \dot{p} \).

**Solution:**

From the first equation of the model, by further differentiation we have

\[
\frac{d^2 \dot{p}}{dt^2} = -\alpha \frac{dU}{dt} - \delta \frac{d^2 U}{dt^2}
\]

In discrete time, this should involve a second difference of price on the left side and a second difference of the rate of unemployment on the right side:

\[
\Delta^2 \dot{p}_t = \Delta (\Delta \dot{p}_t) = \Delta (\dot{p}_{t+2} - \dot{p}_{t+1}) = (\dot{p}_{t+2} - \dot{p}_{t+1}) - (\dot{p}_{t+1} - \dot{p}_t) = \dot{p}_{t+2} - 2 \dot{p}_{t+1} + \dot{p}_t
\]
\[ \Delta^2 U_t = \Delta(U_{t+1} - U_t) = (U_{t+2} - U_{t+1}) - (U_{t+1} - U_t) = U_{t+2} - 2U_{t+1} + U_t \]

The equation in its discrete form becomes
\[ \dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = -\alpha(U_{t+1} - U_t) - \delta(U_{t+2} - 2U_{t+1} + U_t) \]

where from the second equation of the model we have in discrete time
\[ U_{t+1} - U_t = -\beta(m - \dot{p}_t) \] and also
\[ U_{t+2} - 2U_{t+1} + U_t = \beta(\dot{p}_{t+1} - \dot{p}_t) \]

Therefore, the equation for inflation becomes
\[ \dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = \alpha\beta(m - \dot{p}_t) - \beta\delta(\dot{p}_{t+1} - \dot{p}_t) \]
\[ \dot{p}_{t+2} - (2 - \beta\delta)\dot{p}_{t+1} + (1 + \alpha\beta - \beta\delta)\dot{p}_t = \alpha\beta m \]

The equilibrium value for \( \dot{p} \) is \( \bar{p} = \frac{\alpha\beta m}{1 - 2 + \beta\delta + 1 + \alpha\beta - \beta\delta} = \bar{m} \), which we have obtained previously. Analyzing the characteristic roots,
\[ a_1 + a_2 = -b_1 = 2 - \beta\delta \] and
\[ a_1a_2 = b_2 = 1 + \alpha\beta - \beta\delta \] and, therefore,
\[ (1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1a_2 = 1 - 2 + \beta\delta + 1 + \alpha\beta - \beta\delta = \alpha\beta > 0 \]

The last result implies that the characteristic roots can both be bigger than 1 or smaller than 1. This means that a convergent time path for inflation is not impossible. The condition \( 0 < 1 + \alpha\beta - \beta\delta < 1 \) ensures the dynamic stability of inflation.

59. For the discrete-time model in the previous problem, assume the difference
\[ U_{t+1} - U_t = -\beta(m - \dot{p}_{t+1}) \] where the change in unemployment depends on current inflation. How do the results differ from those in the previous problem?

Solution:

The equation of inflation is still
\[ \dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = -\alpha(U_{t+1} - U_t) - \delta(U_{t+2} - 2U_{t+1} + U_t) \]

where \( U_{t+1} - U_t = -\beta(m - \dot{p}_{t+1}) \) and also
\[ U_{t+2} - 2U_{t+1} + U_t = \beta(\dot{p}_{t+1} - \dot{p}_t) \]

Substituting in the first equation,
\[ \dot{p}_{t+2} - 2\dot{p}_{t+1} + \dot{p}_t = \alpha\beta(m - \dot{p}_{t+1}) - \beta\delta(\dot{p}_{t+1} - \dot{p}_t) \]
\[ \dot{p}_{t+2} - (2 - \alpha\beta - \beta\delta)\dot{p}_{t+1} + (1 - \beta\delta)\dot{p}_t = \alpha\beta m \]

The equilibrium value for \( \dot{p} \) is \( \bar{p} = \frac{\alpha\beta m}{1 - 2 + \alpha\beta + \beta\delta + 1 - \beta\delta} = \bar{m} \), which we have obtained previously. For the characteristic roots, we have
\[ a_1 + a_2 = -b_1 = 2 - \alpha\beta - \beta\delta \] and
\[ a_1a_2 = b_2 = 1 - \beta\delta . \]
\[ (1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1a_2 = 1 - 2 + \alpha\beta + \beta\delta + 1 - \beta\delta = \alpha\beta > 0 \]

The last result again shows that a convergent time path for inflation is not impossible. However, this depends on the exact values of the parameters. In this sense, the results are similar to those in the previous model. Furthermore, we see that \( 1 - \beta\delta \) could be less than 1, given the positive values of the parameters, which also allows for convergence.
60. If the extended inflation-unemployment model in its continuous-time form is
\[
\begin{align*}
dp &= -\alpha(U - U_t) \\
dU &= -\beta(m - \dot{p}) - \sigma \frac{dp}{dt}
\end{align*}
\]
\[\alpha > 0, \beta, \sigma > 0\]
modify the model in a discrete-time form and analyze the characteristic roots for inflation \( \dot{p} \). How does inflation in the present or previous period affect unemployment?

Solution:
From the first equation of the model, by further differentiation we obtained
\[\frac{d^2p}{dt^2} = -\alpha \frac{dU}{dt}\]
In discrete time, this should involve a second difference of price on the left side, or
\[
\Delta^2 \dot{p}_t = \Delta(\Delta \dot{p}_t) = \Delta(\dot{p}_{t+1} - \dot{p}_t) = (\dot{p}_{t+2} - \dot{p}_{t+1}) - (\dot{p}_{t+1} - \dot{p}_t) = \dot{p}_{t+2} - 2 \dot{p}_{t+1} + \dot{p}_t
\]
The equation in its discrete form becomes
\[\dot{p}_{t+2} - 2 \dot{p}_{t+1} + \dot{p}_t = -\alpha(U_{t+1} - U_t)\]
The second equation translates into
\[U_{t+1} - U_t = -\beta(m - \dot{p}_t) - \sigma(\dot{p}_{t+1} - \dot{p}_t)\]
Thus, the new model becomes
\[\dot{p}_{t+2} - 2 \dot{p}_{t+1} + \dot{p}_t = -\alpha(U_{t+1} - U_t)\]
\[U_{t+1} - U_t = -\beta(m - \dot{p}_t) - \sigma(\dot{p}_{t+1} - \dot{p}_t)\]
Substituting the difference term for unemployment gives a second-order difference equation in \( \dot{p} \):
\[\dot{p}_{t+2} - 2 \dot{p}_{t+1} + \dot{p}_t = \alpha \beta (m - \dot{p}_t) + \alpha \sigma (\dot{p}_{t+1} - \dot{p}_t)\]
\[\dot{p}_{t+2} - (2 + \alpha \sigma) \dot{p}_{t+1} + (1 + \alpha \beta + \alpha \sigma) \dot{p}_t = \alpha \beta m\]
The equilibrium value for \( \dot{p} \) is
\[\bar{p} = \frac{\alpha \beta m}{1 - 2 - \alpha \sigma + 1 + \alpha \beta + \alpha \sigma} = \dot{m}\]
For the characteristic roots, we have
\[a_1 + a_2 = -b_1 = 2 + \alpha \sigma \quad \text{and} \quad a_1 a_2 = b_2 = 1 + \alpha \beta + \alpha \sigma\]
\[(1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1 a_2 = 1 - 2 - \alpha \sigma + 1 + \alpha \beta + \alpha \sigma = \alpha \beta > 0\]
Here, since \(1 + \alpha \beta + \alpha \sigma\) cannot be between 0 and 1, the roots cannot both be fractions. Therefore, the time path of inflation would not be dynamically stable. Note that, if a different assumption is made about unemployment, such as \(U_{t+1} - U_t = -\beta(m - \dot{p}_{t+1}) - \sigma(\dot{p}_{t+1} - \dot{p}_t)\), the equation becomes
\[\dot{p}_{t+2} - 2 \dot{p}_{t+1} + \dot{p}_t = \alpha \beta (m - \dot{p}_{t+1}) + \alpha \sigma (\dot{p}_{t+1} - \dot{p}_t)\]
\[\dot{p}_{t+2} - (2 - \alpha \sigma) \dot{p}_{t+1} + (1 + \alpha \sigma) \dot{p}_t = \alpha \beta m\]
The intertemporal equilibrium for \( \dot{p} \) is
\[\bar{p} = \frac{\alpha \beta m}{1 - 2 + \alpha \beta - \alpha \sigma + 1 + \alpha \sigma} = \dot{m}\]
For characteristic roots, we have
\[a_1 + a_2 = -b_1 = 2 - \alpha \beta + \alpha \sigma \quad \text{and} \quad a_1 a_2 = b_2 = 1 + \alpha \sigma\]
\[(1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1 a_2 = 1 - 2 + \alpha \beta - \alpha \sigma + 1 + \alpha \sigma = \alpha \beta > 0\]
Here since $1 + \alpha \sigma$ cannot be between 0 and 1, the roots cannot both be fractions. Therefore, the time path of inflation would not be dynamically stable.

61. Transform the national-income model presented in problem 35 from continuous into discrete time. Solve for aggregate output $Y$, and find its intertemporal equilibrium.

Solution:
We recall that in its continuous form the model is:

\[
\dot{p} = \pi + \alpha(Y - \bar{Y}) \quad \alpha > 0 \quad \text{(Phillips relation)}
\]
\[
\frac{d\pi}{dt} = j(\dot{p} - \pi) \quad 0 < j \leq 1 \quad \text{(adaptive expectations)}
\]
\[
m - \dot{p} = \beta \frac{dY}{dt} - \delta \frac{d\pi}{dt} \quad \beta, \delta > 0 \quad \text{($LM$ schedule)}
\]

Solution:
In discrete time, the model can be written as:

\[
\dot{p}_t = \pi_t + \alpha(Y_t - \bar{Y})
\]
\[
\pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t)
\]
\[
m - \dot{p}_t = \beta(Y_{t+1} - Y_t) - \delta(\pi_{t+1} - \pi_t)
\]

From the first two equations,
\[
\pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) = \alpha j(Y_t - \bar{Y})
\]

We directly substitute the difference for $\pi_t$ in the third equation:
\[
m - \dot{p}_t = \beta(Y_{t+1} - Y_t) - \alpha \delta j(Y_t - \bar{Y})
\]

We have one $\dot{p}_t$ to get rid of in order to obtain a difference equation in $Y$ solely. Extending the first equation by one time period gives:

\[
\dot{p}_{t+1} = \pi_{t+1} + \alpha(Y_{t+1} - \bar{Y})
\]
\[
\dot{p}_t = \pi_t + \alpha(Y_t - \bar{Y})
\]

and expressing the difference $\Delta p_t$,
\[
\dot{p}_{t+1} - \dot{p}_t = \pi_{t+1} - \pi_t + \alpha(Y_{t+1} - Y_t) = \alpha j(Y_t - \bar{Y}) + \alpha(Y_{t+1} - Y_t)
\]

Furthermore, from the equation for aggregate output,
\[
m - \dot{p}_t = \beta(Y_{t+1} - Y_t) - \alpha \delta j(Y_t - \bar{Y})
\]
\[
m - \dot{p}_{t+1} = \beta(Y_{t+2} - Y_{t+1}) - \alpha \delta j(Y_{t+1} - \bar{Y})
\]

where we subtract the two equations
\[
\dot{p}_{t+1} - \dot{p}_t = \beta(2Y_{t+1} - Y_{t+2} - Y_t) - \alpha \delta j(Y_t - Y_{t+1})
\]

But for the difference $\Delta p_t$, we already have
\[
\dot{p}_{t+1} - \dot{p}_t = \alpha j(Y_t - \bar{Y}) + \alpha(Y_{t+1} - Y_t)
\]

Thus, equating the two,
\[ \alpha j(Y_t - \bar{Y}) + \alpha(Y_{t+1} - Y_t) = \beta(2Y_{t+1} - Y_{t+2} - Y_t) - \alpha \delta j(Y_t - Y_{t+1}) \]

which is a second-order differential equation solely in \( Y \). Rearranging and normalizing leads to

\[ Y_{t+2} + \frac{(\alpha - 2\beta - \alpha \delta j)}{\beta} Y_{t+1} + \frac{(\alpha j - \alpha + \beta + \alpha \delta j)}{\beta} Y_t = \frac{\alpha j \bar{Y}}{\beta} \]

Intertemporal equilibrium output is

\[ Y_p = \frac{c}{1 + b_1 + b_2} = \frac{\alpha j \bar{Y}}{\beta \left[ 1 + \frac{\alpha - 2\beta - \alpha \delta j}{\beta} + \frac{\alpha j - \alpha + \beta + \alpha \delta j}{\beta} \right]} = \bar{Y} \]

The intertemporal equilibrium is exactly equal to the full-employment output, which represents a static equilibrium.

62. Convert the national-income model given in problem 35 from continuous into discrete time and solve for actual inflation \( \dot{p} \). Prove that its time path is the same as that for aggregate output \( Y \).

**Solution:**

We convert the model into

\[ \dot{p}_t = \pi_t + \alpha(Y_t - \bar{Y}) \]
\[ \pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) \]
\[ \dot{m} - \dot{p}_t = \beta(Y_{t+1} - Y_t) - \delta(\pi_{t+1} - \pi_t) \]

Extending the first equation by one time period gives

\[ \dot{p}_{t+1} = \pi_{t+1} + \alpha(Y_{t+1} - \bar{Y}) \]
\[ \dot{p}_t = \pi_t + \alpha(Y_t - \bar{Y}) \]

and expressing the difference \( \Delta p_t \),

\[ \dot{p}_{t+1} - \dot{p}_t = \pi_{t+1} - \pi_t + \alpha(Y_{t+1} - Y_t) = j(\dot{p}_t - \pi_t) + \alpha(Y_{t+1} - Y_t) \]

From the last equation of the model,

\[ \dot{m} - \dot{p}_t = \beta(Y_{t+1} - Y_t) - \delta j(\dot{p}_t - \pi_t) \]

and expressing the term \( (Y_{t+1} - Y_t) \),

\[ Y_{t+1} - Y_t = \frac{1}{\beta} [\dot{m} - \dot{p}_t + \delta j(\dot{p}_t - \pi_t)] \]

Substituting this term in the difference equation for inflation,

\[ \dot{p}_{t+1} - \dot{p}_t = j(\dot{p}_t - \pi_t) + \frac{\alpha}{\beta} (\dot{m} - \dot{p}_t) + \frac{\alpha \delta j}{\beta} (\dot{p}_t - \pi_t) \]
\[ \dot{p}_{t+1} - \dot{p}_t = \frac{(\beta j + \alpha \delta j)}{\beta} (\dot{p}_t - \pi_t) + \frac{\alpha}{\beta} (\dot{m} - \dot{p}_t) \]

Extending this by one time period,

\[ \dot{p}_{t+2} - \dot{p}_{t+1} = \frac{(\beta j + \alpha \delta j)}{\beta} (\dot{p}_{t+1} - \pi_{t+1}) + \frac{\alpha}{\beta} (\dot{m} - \dot{p}_{t+1}) \]

and subtracting the two equations,
\[ 2 \dot{p}_{t+1} - \dot{p}_{t+2} - \dot{p}_t = \frac{(\beta j + \alpha \delta j)}{\beta} (\ddot{p}_t - \dot{p}_{t+1} + \pi_{t+1} - \pi_t) + \frac{\alpha}{\beta} (\dot{p}_{t+1} - \dot{p}_t) \]

\[ 2 \dot{p}_{t+1} - \dot{p}_{t+2} - \dot{p}_t = \frac{\alpha \delta j - \alpha}{\beta} (\ddot{p}_t - \dot{p}_{t+1}) + \frac{\alpha}{\beta} (\dot{m} - \dot{p}_t) \]

Substituting the last term in the equation for inflation,

\[ 2 \dot{p}_{t+1} - \dot{p}_{t+2} - \dot{p}_t = \frac{(\beta j + \alpha \delta j - \alpha)}{\beta} (\ddot{p}_t - \dot{p}_{t+1}) + \frac{\alpha j}{\beta} (\dot{m} - \dot{p}_t) \]

Rearranging,

\[ 2 \dot{p}_{t+1} - \dot{p}_{t+2} - \dot{p}_t = \frac{\alpha \delta j - \alpha}{\beta} (\ddot{p}_t - \dot{p}_{t+1}) + \frac{\alpha j}{\beta} (\dot{m} - \dot{p}_t) \]

\[ \dot{p}_{t+2} + \frac{\alpha \delta j - 2 \beta}{\beta} \dot{p}_{t+1} + \frac{\beta + \alpha \delta j - \alpha + \alpha j}{\beta} \dot{p}_t = \frac{\alpha jm}{\beta} \]

The coefficients \( b_1 \) and \( b_2 \) are the same as those in the differential equation from aggregate output \( Y \) obtained in the previous problem. This indicates that the two variables have the same time path.

Furthermore, for the equilibrium level of inflation,

\[ \bar{\pi} = \frac{c}{1 + b_1 + b_2} = \frac{\alpha jm}{\beta \left(1 + \frac{\alpha \delta j - 2 \beta + \beta + \alpha \delta j - \alpha + \alpha j}{\beta}\right)} = \bar{m} \]

63. In the discrete-time national-income model, assume that the government sets the growth of money supply based on the increase in national income from the previous period. Thus, the model becomes

\[ \dot{p}_t = \pi_t + \alpha (Y_t - \bar{Y}) \]

\[ \pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) \]

\[ \dot{m} - \dot{p}_t = \beta(Y_t - Y_{t-1}) - \delta(\pi_{t+1} - \pi_t) \]

Solve the model for expected inflation \( \pi \).

**Solution:**

We rewrite the last equation

\[ \dot{m} - \dot{p}_{t+1} = \beta(Y_{t+1} - Y_t) - \delta(\pi_{t+2} - \pi_{t+1}) \]

From the first two equations,

\[ \pi_{t+1} - \pi_t = j(\dot{p}_t - \pi_t) = \alpha j(Y_t - \bar{Y}) \]

Extending this equation by a period and subtracting the two equations,

\[ \pi_{t+2} - \pi_{t+1} = \alpha j(Y_{t+1} - \bar{Y}) \]

\[ \pi_{t+1} - \pi_t = \alpha j(Y_t - \bar{Y}) \]

\[ \pi_{t+2} - 2\pi_{t+1} + \pi_t = \alpha j(Y_{t+1} - Y_t) \]

and substituting the term \( Y_{t+1} - Y_t \) in the equation for expected inflation,

\[ \dot{m} - \dot{p}_{t+1} = \frac{\beta}{\alpha j} (\pi_{t+2} - 2\pi_{t+1} + \pi_t) - \delta(\pi_{t+2} - \pi_{t+1}) \]
From the second equation, we express \( \dot{p}_t \):

\[
\dot{p}_t = \frac{1}{j} \pi_{t+1} + \frac{(j-1)}{j} \pi_t
\]

which corresponds to

\[
\dot{p}_{t+1} = \frac{1}{j} \pi_{t+2} + \frac{(j-1)}{j} \pi_{t+1}
\]

and substituting again

\[
m - \frac{1}{j} \pi_{t+2} - \frac{j-1}{j} \pi_{t+1} = \frac{\beta}{\alpha j} (\pi_{t+2} - 2\pi_{t+1} + \pi_t) - \delta (\pi_{t+2} - \pi_{t+1})
\]

\[
\alpha j \dot{m} - \alpha \pi_{t+2} - \alpha (j-1) \pi_{t+1} = \beta (\pi_{t+2} - 2\pi_{t+1} + \pi_t) - \alpha \delta j (\pi_{t+2} - \pi_{t+1})
\]

Rearranging and normalizing,

\[
(\alpha + \beta - \alpha \delta j)\pi_{t+2} + \left[\alpha (j-1) - 2\beta + \alpha \delta j\right] \pi_{t+1} + \beta \pi_t = \alpha j \dot{m}
\]

\[
\pi_{t+2} + \left[\alpha (j-1) - 2\beta + \alpha \delta j\right] \pi_{t+1} + \frac{\beta}{(\alpha + \beta - \alpha \delta j)} \pi_t = \frac{\alpha j \dot{m}}{(\alpha + \beta - \alpha \delta j)}
\]

Hence, the equilibrium value for expected inflation is

\[
\bar{\pi} = \frac{\alpha j \dot{m}}{1 + b_1 + b_2} = \frac{\alpha j \dot{m}}{(\alpha + \beta - \alpha \delta j) \left(1 + \frac{\alpha j - \alpha - 2\beta + \alpha \delta j + \beta}{\alpha + \beta - \alpha \delta j}\right)} = \dot{m}
\]

64. A market equilibrium model is given such that the demand and supply functions for a commodity are

\[
q^d_i = 6 - 5p_i
\]

\[
q^s_i = -4 + 15p_i
\]

Furthermore, it is known that the increase in market price from one period to another depends on excess demand by the adjustment coefficient 1/5 such that \( p_{t+1} - p_t = \frac{1}{5} (q^d_i - q^s_i) \). Solve the model with the help of a second-order difference equation. Find the general solution and intertemporal equilibrium for \( p_t \). Then, using the steps of the Cobweb model, form a first-order difference equation and solve again for market equilibrium. Compare the time paths and equilibrium values of price \( p_t \) using the two approaches.

**Solution:**

Substituting the demand and the supply function in the last equation,

\[
p_{t+1} - p_t = \frac{1}{5} (6 - 5p_t + 4 - 15p_{t-1})
\]

\[
p_{t+1} - p_t = 2 - p_t - 3p_{t-1}
\]

\[
p_{t+1} + 3p_{t-1} = 2
\]

which transforms into

\[
p_{t+2} + 3p_t = 2
\]

where \( b_1 = 0 \) and \( b_2 = 3 \)

Finding intertemporal equilibrium price,

\[
\bar{p} = \frac{2}{1 + 3} = \frac{1}{2}
\]

For the characteristic roots we have \( a^2 + 3 = 0 \) and \( a = \pm \sqrt{3} \); or, we have the complex-root case. This implies that the time path of price would be stepped fluctuation and divergent from the
equilibrium of 1/2 since the absolute value of both roots is greater than 1, that is, $\sqrt{3} > 1$. The general solution for price could be written as

$$p_t = A_1 \left(-i\sqrt{3}\right) + A_2 \left(i\sqrt{3}\right) + \frac{1}{2}$$

Alternatively, we can use the steps of the Cobweb model according to which at any point in time the market clears. This is different from the previous approach where we assumed that the market may not always be in equilibrium and actually diverges from this equilibrium as $t \to \infty$. Therefore,

$$q_t^d = q_t^s$$

$$6 - 5p_t = -4 + 15p_{t-1}$$

$$p_t + 3p_{t-1} = 2$$

This equation resembles the one previously obtained; but unlike it, it is a first-order difference equation. Solving through the well-known method,

$$p_t = \left(p_o - \frac{2}{1+3}\right)\left(-3\right) + \frac{2}{1+3}$$

$$p_t = \left(p_o - \frac{1}{2}\right)\left(-3\right) + \frac{1}{2}$$

Again, an equilibrium value of 1/2 is obtained, and the time path is oscillatory and divergent since $-3 < 0$ and $3 > 1$. Thus, the results of the two approaches are pretty similar — in both cases, price diverges from the intertemporal equilibrium of $1/2$, although in the first case the fluctuation is stepped and in the second it is ordinary oscillation.

65. In a market for a given commodity, supply depends on price in the previous period but also on the increase in price from the previous to the current period. Thus, as producers see price rising, they feel stimulated to supply more. The demand and supply functions are

$$q_t^d = \alpha - \beta p_t$$

$$q_t^s = -\gamma + \delta p_{t-1} + \varepsilon(p_t - p_{t-1})$$

$$p_{t+1} - p_t = j(q_t^d - q_t^s)$$

where all parameters are positive. Solve the model with the help of a second-order difference equation. Then assume the market is constantly in equilibrium, and solve using a first-order difference equation. Compare the two equilibrium values.

**Solution:**

Substituting the demand and the supply function in the last equation,

$$p_{t+1} - p_t = j(\alpha - \beta p_t + \gamma - \delta p_{t-1} - \varepsilon p_t + \varepsilon p_{t-1})$$

$$p_{t+1} - \left[1 - (\beta + \varepsilon)j\right]p_t + (\delta - \varepsilon)jp_{t-1} = j(\alpha + \gamma)$$

Extending further by one time period,

$$p_{t+2} - \left[1 - (\beta + \varepsilon)j\right]p_{t+1} + (\delta - \varepsilon)jp_t = j(\alpha + \gamma)$$

Finding equilibrium price,
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\[ \bar{p} = \frac{j(\alpha + \gamma)}{1 - 1 + (\beta + \varepsilon)j + (\delta - \varepsilon)j} = \frac{\alpha + \gamma}{\beta + \delta} \]

Assuming the market always clears, we equate demand and supply:

\[ \alpha - \beta p_t = -\gamma + \delta p_{t-1} + \varepsilon(p_t - p_{t-1}) \]

\[ (\beta + \varepsilon)p_t + (\delta - \varepsilon)p_{t-1} = \alpha + \gamma \]

Normalizing the equation,

\[ p_t + \frac{(\delta - \varepsilon)}{\beta + \varepsilon}p_{t-1} = \frac{\alpha + \gamma}{\beta + \varepsilon} \]

\[ p_t = \left[ p_o - \frac{\alpha + \gamma}{(\beta + \varepsilon)\left(1 + \frac{\delta - \varepsilon}{\beta + \varepsilon}\right)} \right] \left( -\frac{\delta - \varepsilon}{\beta + \varepsilon} \right)^t + \frac{\alpha + \gamma}{\beta + \varepsilon} \]

Thus, the two equilibrium values are the same.

66. Assume a market for a commodity in which producers continuously follow price trends and their decision to supply presently depends on price levels in two consecutive previous periods. Thus, the model becomes

\[ q_t^d = \alpha - \beta p_t \]

\[ q_t^s = -\gamma + \delta p_{t-1} + \sigma p_{t-2} \]

Assume the market clears at any point in time and all parameters are positive. Find the intertemporal equilibrium price and quantity.

Solution:

Equating demand with supply,

\[ \alpha - \beta p_t = -\gamma + \delta p_{t-1} + \sigma p_{t-2} \]

\[ \beta p_t + \delta p_{t-1} + \sigma p_{t-2} = \alpha + \gamma \]

Normalizing and extending further by two time periods,

\[ p_{t+2} + \frac{\delta}{\beta} p_{t+1} + \frac{\sigma}{\beta} p_t = \frac{\alpha + \gamma}{\beta} \]

Finding equilibrium price,

\[ \bar{p} = \frac{(\alpha + \gamma)}{\beta + \delta + \sigma} = \frac{\alpha + \gamma}{\beta + \delta + \sigma} \]

From the demand function,

\[ \bar{q}^d = \alpha - \beta \bar{p} = \alpha - \frac{\beta(\alpha + \gamma)}{\beta + \delta + \sigma} = \frac{\alpha(\delta + \sigma) - \beta\gamma}{\beta + \delta + \sigma} \]
67. Assume a market for a commodity in which consumers expect price to rise in the future so their present demand is positively related to price in the future period. Thus, the demand and supply functions are

\[ q^d_t = \alpha - \beta p_t + \eta p_{t+1} \]
\[ q^s_t = -\gamma + \delta p_t + \sigma p_{t-1} \]

Assume the market always clears and all parameters are positive. Find the intertemporal equilibrium price and quantity. How does a positive or a negative value of the expectations coefficient \( \eta \) affect the value of equilibrium price \( \bar{p} \)?

**Solution:**

Equating demand with supply,

\[ \alpha - \beta p_t + \eta p_{t+1} = -\gamma + \delta p_t + \sigma p_{t-1} \]

\[ \eta p_{t+1} - (\beta + \delta) p_t - \sigma p_{t-1} = -\alpha - \gamma \]

Normalizing and extending by one time period,

\[ p_{t+2} - \frac{(\beta + \delta)}{\eta} p_{t+1} - \frac{\sigma}{\eta} p_t = -\frac{\alpha + \gamma}{\eta} \]

Finding equilibrium price,

\[ \bar{p} = -\frac{\alpha + \gamma}{\eta \left(1 - \frac{\beta + \delta}{\eta} - \frac{\sigma}{\eta}\right)} = -\frac{\alpha + \gamma}{\eta - \beta - \delta - \sigma} = \frac{\alpha + \gamma}{\beta + \delta + \sigma - \eta} \]

From the demand function,

\[ \bar{d} = \alpha - \beta \bar{p} + \eta \bar{p} = \alpha - (\beta - \eta) \bar{p} = \alpha - \frac{(\beta - \eta)(\alpha + \gamma)}{(\beta + \delta + \sigma - \eta)} = \frac{\alpha(\delta + \sigma - \gamma(\beta - \eta))}{(\beta + \delta + \sigma - \eta)} \]

To have a meaningful price, we need \( \beta + \delta + \sigma > \eta \). Note that if \( \eta > 0 \), as the model assumes, consumers expect price to rise in the future, so they buy more presently. Thus, the equilibrium price indeed turns out to be higher than if consumer expectations of price are not taken into account.

However, if hypothetically, \( \eta < 0 \) and, hence, consumers predict that price would fall in the future period, this would reduce their present demand for the good. With \( \eta < 0 \), the equilibrium price of the commodity falls lower. Therefore, we can conclude that consumers’ expectations do in fact shape market price and its movement.

68. In a given market, producers continuously follow price trends and base their decisions on price in two consecutive previous periods. Consumers, on the other hand, are influenced by current price but also by the increase in the price level. Thus, the model is

\[ q^d_t = \alpha - \beta p_t + \eta (p_t - p_{t-1}) \]
\[ q^s_t = -\gamma + \delta p_{t-1} + \sigma p_{t-2} \]

Assume that the market clears at any point in time and that all parameters are positive. Find the intertemporal equilibrium price and check how it depends on the expectations coefficient \( \eta \).

**Solution:**

Equating demand with supply,
\( \alpha - \beta p_t + \eta(p_t - p_{t-1}) = -\gamma + \delta p_{t-1} + \sigma p_{t-2} \)

\( (\eta - \beta)p_t - (\delta + \eta)p_{t-1} - \sigma p_{t-2} = -\alpha - \gamma \)

Normalizing and extending further by two time periods,

\[
p_{t+2} - \frac{\delta + \eta}{\eta - \beta} p_{t+1} - \frac{\sigma}{\eta - \beta} p_t = -\frac{\alpha + \gamma}{\eta - \beta}
\]

Finding equilibrium price,

\[
\bar{p} = -\frac{(\alpha + \gamma)}{(\eta - \beta)\left(1 - \frac{\delta + \eta}{\eta - \beta} - \frac{\sigma}{\eta - \beta}\right)} = -\frac{\alpha + \gamma}{\eta - \beta - \delta - \eta - \sigma} = \frac{\alpha + \gamma}{\beta + \delta + \sigma}
\]

We obtain that intertemporal equilibrium price is independent of the expectations coefficient \( \eta \) according to the assumptions of the model.

69. Recall the nonlinear Cobweb model discussed in chapter 10. Consumers are negatively influenced by current price. Producers, on the other hand, make their decisions based on price in the current and the previous two periods.

\[
q_{t+2}^d = \alpha p_{t+2}^{-\beta}
\]

\[
q_{t+2}^s = \gamma p_{t+2}^\delta p_t
\]

\( \alpha, \beta > 0 \)

\( \gamma, \delta > 0 \)

Solve the model for intertemporal equilibrium price and quantity. Express the characteristic roots. What is the condition for a single root to exist? Is dynamic stability of market price plausible?

Solution:

In equilibrium \( q_{t+2}^d = q_{t+2}^s \) and

\[
\alpha p_{t+2}^{-\beta} = \gamma p_{t+2}^\delta p_t
\]

Taking the natural log of both sides,

\[
\ln \alpha - \beta \ln p_{t+2} = \ln \gamma + \ln p_{t+2} + \delta \ln p_{t+1} + \ln p_t
\]

\[
(\beta + 1) \ln p_{t+2} + \delta \ln p_{t+1} + \ln p_t = \ln \alpha - \ln \gamma
\]

Normalizing,

\[
\ln p_{t+2} + \frac{\delta}{\beta + 1} \ln p_{t+1} + \frac{1}{\beta + 1} \ln p_t = \frac{\ln \alpha}{\beta + 1}
\]

Setting \( y_t = \ln p_t \),

\[
y_{t+2} + \frac{\delta}{\beta + 1} y_{t+1} + \frac{1}{\beta + 1} y_t = \ln \frac{\alpha}{\gamma}
\]

\[
\bar{y} = \frac{\ln \frac{\alpha}{\gamma}}{(\beta + 1)\left(1 + \frac{\delta + 1}{\beta + 1}\right)} = \frac{\ln \frac{\alpha}{\gamma}}{2 + \beta + \delta}
\]

or

\[
\ln \bar{p} = \frac{\ln \frac{\alpha}{\gamma}}{2 + \beta + \delta}
\]

Taking the antilog of both sides,
\[ p = e^{\frac{\ln a}{\gamma}} = \left( e^{\ln a} \right)^{\frac{1}{\gamma}} = \left( \frac{\alpha}{\gamma} \right)^{\frac{1}{\gamma}} \]

From the demand function, we express equilibrium quantity

\[ \bar{q} = \alpha \bar{p}^{-\beta} = \alpha \left( \frac{\alpha}{\gamma} \right)^{\frac{\beta}{2 + \beta + \delta}} = \alpha^{2 + \beta + \delta} \gamma \left( \frac{\beta}{2 + \beta + \delta} \right) \]

If the market cleared at any point in time, we would have

\[ \alpha(\bar{p})^{-\beta} = \gamma(\bar{p})^{\delta+\gamma} \]
\[ \frac{\alpha}{\gamma} = (\bar{p})^{2 + \beta + \delta} \]
\[ \bar{p} = \left( \frac{\alpha}{\gamma} \right)^{\frac{1}{2 + \beta + \delta}} \]

which is the equilibrium value obtained previously. The characteristic roots are

\[ a_{1,2} = \frac{-\delta}{\beta + 1} \pm \frac{\delta^2 - \frac{4}{(\beta + 1)^2}}{2} = -\delta \pm \sqrt{\delta^2 - 4(\beta + 1)} \]

There will be a single real root, if the parameters are such that \( \delta = 2(\beta + 1) \). Furthermore,

\[ a_1 + a_2 = -b_1 = -\frac{\delta}{\beta + 1} \quad \text{and} \quad a_1 a_2 = b_2 = \frac{1}{\beta + 1} \quad \text{so} \quad a_1 a_2 \in (0, 1) \]

\( (1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1 a_2 = 1 + \frac{\delta}{\beta + 1} + \frac{1}{\beta + 1} = 1 + \frac{\delta + 1}{\beta + 1} > 1 \)

Since the parameters \( \beta \) and \( \delta \) are positive by definition, we have \( (1 - a_1)(1 - a_2) > 1 \) and both \( a_1 \) and \( a_2 \) are negative (why?), we may have \( |a_1| > 1 \) and \( |a_2| > 1 \). Therefore, dynamic stability for market price is plausible.

70. Assume that the nonlinear Cobweb model describes the electricity sector. Consumers are negatively influenced by rises in electricity price in three consecutive periods. They have switched to alternative energy sources in the previous periods, so they are most elastic at present. The electrical company, on the other hand, made a huge investment in period \( t \), so supply was most elastic then. The company is stimulated by a higher price in each period.

\[ q_{i+2}^d = \alpha P_{i+2}^{-\beta} \quad \alpha, \beta > 0 \]
\[ q_{i+2}^s = \gamma P_{i+2} P_{i+1}^\delta \quad \gamma, \delta > 0 \]

Solve the model for intertemporal equilibrium price and quantity. Express the characteristic roots. What is the condition for complex roots to exist?

**Solution:**

In equilibrium, \( q_{i+2}^d = q_{i+2}^s \) and

\[ \alpha P_{i+2}^{-\beta} = \gamma P_{i+2} P_{i+1}^\delta \]

\[ P_{i+2} = \left( \frac{\alpha}{\gamma} \right)^{\frac{1}{2 + \beta + \delta}} \]
Taking the natural log of both sides,
\[
\ln \alpha - \beta \ln p_{t+2} - \ln p_{t+1} - \ln p_t = \ln \gamma + \ln p_{t+2} + \ln p_{t+1} + \delta \ln p_t
\]
\[
(\beta + 1) \ln p_{t+2} + 2 \ln p_{t+1} + (\delta + 1) \ln p_t = \ln \alpha - \ln \gamma
\]

Normalizing,
\[
\ln p_{t+2} + \frac{2}{\beta + 1} \ln p_{t+1} + \frac{\delta + 1}{\beta + 1} \ln p_t = \ln \frac{\alpha}{\gamma}
\]

Setting \( y_t = \ln p_t \),
\[
y_{t+2} + \frac{2}{\beta + 1} y_{t+1} + \frac{\delta + 1}{\beta + 1} y_t = \ln \frac{\alpha}{\gamma}
\]
\[
\bar{y} = \frac{\ln \frac{\alpha}{\gamma}}{(\beta + 1) \left(1 + \frac{2 + \delta + 1}{\beta + 1}\right)} = \ln \frac{\alpha}{\gamma} \quad \text{or}
\]
\[
\ln \bar{p} = \ln \frac{\alpha}{\gamma} \quad \frac{4 + \beta + \delta}{4 + \beta + \delta}
\]

Taking the antilog of both sides,
\[
\bar{p} = e^{\frac{\ln \frac{\alpha}{\gamma}}{4 + \beta + \delta}} = (e^{\ln \frac{\alpha}{\gamma}})^{\frac{1}{4 + \beta + \delta}} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{4 + \beta + \delta}}
\]

From the demand function, we express equilibrium quantity
\[
\bar{q} = \alpha \bar{p}^{\beta - 2} = \alpha \left(\frac{\alpha}{\gamma}\right)^{\frac{\beta + 2}{4 + \beta + \delta}} = \alpha \left(\frac{2 + \delta}{4 + \beta + \delta}\right)^{\beta + 2} \left(\frac{\beta + 2}{4 + \beta + \delta}\right)^{\gamma + \beta + \delta}
\]

If the market was constantly in equilibrium, we would have
\[
\alpha(\bar{p})^{\beta - 2} = \gamma(\bar{p})^{\delta + 2}
\]
\[
\frac{\alpha}{\gamma} = (\bar{p})^{\frac{4 + \beta + \delta}{4 + \beta + \delta}} \quad \text{and}
\]
\[
\bar{p} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{4 + \beta + \delta}}
\]

which confirms our previous result. For the characteristic roots,
\[
a_{1,2} = \frac{-2}{\beta + 1} \pm \frac{\sqrt{4 - (\beta + 1)(\delta + 1)}}{2(\beta + 1)}
\]

For complex roots to obtain, we need \((\beta + 1)(\delta + 1) > 4\).